What is integrability of (discrete) variational systems?

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Discretization in Geometry and Dynamics SFB Transregio 109

Based on:

1. Yu. S. *Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms*. J. Geom. Mechanics, 2013, **5**, No. 3, p. 365–379.

2. R. Boll, M. Petrera, Yu. S. *What is integrability of discrete variational systems?* Proc. Royal Soc. A, 2014, **470**, No. 2162, 20130550, 15 pp.

3. A. Bobenko, Yu. S. *Discrete pluriharmonic functions as solutions of linear pluri-Lagrangian systems*, Commun. Math. Phys. 2015, **336**, No. 1, p. 199–215.

4. Yu. S., M. Vermeeren. *On the Lagrangian structure of integrable hierarchies*, `arXiv:1510.03724 [math-ph]`.

Main question

According to the spirit of Geometric Numerical Integration, use *variational integrators* to discretize Lagrangian systems.

But how should one take into account *integrability* of the original system?

And what does it mean for a Lagrangian system (continuous or discrete) to be integrable?

Recent development started with:

- S. Lobb, F.W. Nijhoff. *Lagrangian multiforms and multidimensional consistency*, J. Phys. A, 2009, **42**, 4540103

who discovered closedness of Lagrangian 2-form on solutions of systems of quad-equations (for a part of ABS list).

More remote precursors

- *Theory of pluriharmonic functions:* $f : \mathbb{C}^m \to \mathbb{R}$ minimizes the Dirichlet functional $E_\Gamma = \int_\Gamma |(f \circ \Gamma)_z|^2 dz \wedge d\bar{z}$ along any holomorphic curve $\Gamma : \mathbb{C} \to \mathbb{C}^m$.
- *Baxter's Z-invariance* of solvable models of statistical mechanics: invariance of the partition function under elementary local transformations of the underlying graph.
- *Variational symmetries* (classical work by E. Noether).

Pluri-Lagrangian problem, discrete time, $d = 1$

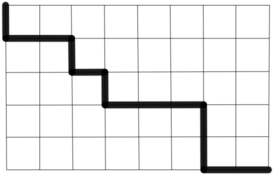
A *discrete 1-form* $\mathcal{L}$ is a skew-symmetric function on directed edges of $\mathbb{Z}^m$, depending on a field $x : \mathbb{Z}^m \to \mathcal{X}$ (where $\mathcal{X}$ is a vector space):

$$\mathcal{L}(n, n+e_i) = \Lambda_i(x, x_i) \quad \Leftrightarrow \quad \mathcal{L}(n, n-e_i) = -\Lambda_i(x_{-i}, x).$$

A *discrete curve* Γ in $\mathbb{Z}^m$ is a concatenation of edges $\mathfrak{e}_k$.

Action functional along Γ:

$$S_\Gamma = \sum_{k \in \mathbb{Z}} \mathcal{L}(\mathfrak{e}_k).$$



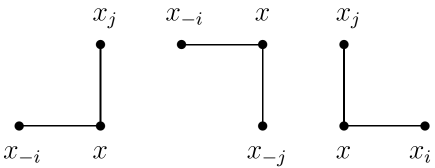
Problem. Find functions $x : \mathbb{Z}^m \to \mathcal{X}$ delivering critical points for the functional S_Γ along *any* discrete curve Γ in $\mathbb{Z}^m$.

2D corner equations

Theorem. Function $x : \mathbb{Z}^m \to \mathcal{X}$ solves the pluri-Lagrangian problem for $\mathcal{L}(n, n+e_i) = \Lambda_i(x, x_i)$, iff the following *2D corner equations* are satisfied:

$$\begin{aligned}
\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x} + \frac{\partial \Lambda_j(x, x_j)}{\partial x} &= 0, \\
\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x} - \frac{\partial \Lambda_j(x_{-j}, x)}{\partial x} &= 0, \\
\frac{\partial \Lambda_i(x, x_i)}{\partial x} - \frac{\partial \Lambda_j(x, x_j)}{\partial x} &= 0
\end{aligned}$$

(four equations per elementary square σ_{ij}).

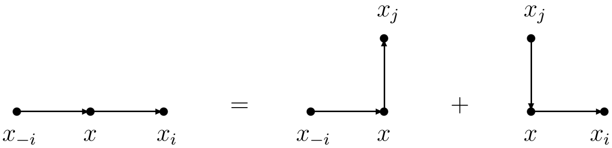


Consistency of 2D corner equations

Definition. System of 2D corner equations is called *consistent*, if, for any square σ_{ij}, it has the minimal possible rank 2, i.e., if exactly two of these four equations are independent.

Remark. Standard (single-time) discrete EL equations are *consequences* of 2D corner equations:

$$\frac{\partial \Lambda_i(x_{-i}, x)}{\partial x} + \frac{\partial \Lambda_i(x, x_i)}{\partial x} = 0,$$



Commuting symplectic maps

If 2D corner equations are satisfied, then there exists $p : \mathbb{Z}^m \to T^*\mathcal{X}$ satisfying

$$p = \frac{\partial \Lambda_i(x, x_i)}{\partial x} = -\frac{\partial \Lambda_j(x_{-j}, x)}{\partial x}, \quad i, j = 1, \ldots, m.$$

Theorem. 2D corner equations are consistent, iff symplectic maps $F_i : (x, p) \mapsto (x_i, p_i)$ defined by

$$p = \frac{\partial \Lambda_i(x, x_i)}{\partial x}, \quad p_i = -\frac{\partial \Lambda_i(x, x_i)}{\partial x_i},$$

commute, $F_i \circ F_j = F_j \circ F_i$.

Almost closedness of multi-time 1-form

Theorem. The following identities hold true on solutions of multi-time discrete Euler-Lagrange equations:

$$d\mathcal{L}(\sigma_{ij}) = \Lambda_i(x, x_i) + \Lambda_j(x_i, x_{ij}) - \Lambda_i(x_j, x_{ij}) - \Lambda_j(x, x_j) = \ell_{ij} = \mathrm{const}.$$

Proof. Partial derivatives of the left-hand side with respect to each of x, x_i, x_j and x_{ij} vanish on solutions, according to the corner equations. ■

In particular, if all these constants ℓ_{ij} vanish, then the discrete 1-form $\mathcal{L}$ is closed on solutions, so that the extremal value of the action functional S_Γ does not depend on the choice of the curve Γ connecting two given points in $\mathbb{Z}^m$.

Closedness of multi-time 1-form vs. spectrality

Let $F_\lambda : (x, p) \mapsto (\widetilde{x}, \widetilde{p})$ be a *1-parameter family* of commuting symplectic maps (*Bäcklund transformations*), with Lagrange function $\Lambda(x, \widetilde{x}; \lambda)$.

For a second such map, we write $F_\mu : (x, p) \mapsto (\widehat{x}, \widehat{p})$.

From previous theorem:

$$\Lambda(x, \widetilde{x}; \lambda) + \Lambda(\widetilde{x}, \widehat{\widetilde{x}}; \mu) - \Lambda(x, \widehat{x}; \mu) - \Lambda(\widehat{x}, \widehat{\widetilde{x}}; \lambda) = \ell(\lambda, \mu).$$

Theorem. The discrete 1-form $\mathcal{L}$ is closed on solutions of the multi-time Euler-Lagrange equations iff

$$\partial\Lambda(x, \widetilde{x}; \lambda)/\partial\lambda$$

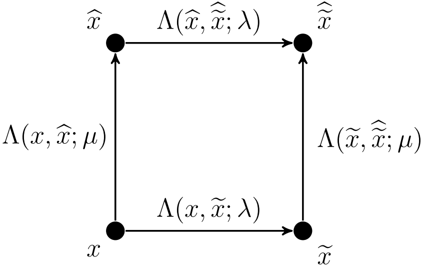
is a common integral of motion for all F_μ (*spectrality*; discovered on examples by Kuznetsov-Sklyanin (1998)).

Closedness of multi-time 1-form vs. spectrality (continued)

Proof. Due to skew-symmetry, $\ell(\lambda,\mu)=0$ is equivalent to $\partial\ell(\lambda,\mu)/\partial\lambda=0$, that is, to

$$\frac{\partial\Lambda(x,\widetilde{x};\lambda)}{\partial\lambda}-\frac{\partial\Lambda(\widehat{x},\widehat{\widetilde{x}};\lambda)}{\partial\lambda}=0.$$

This is equivalent to saying that $\partial\Lambda(x,\widetilde{x};\lambda)/\partial\lambda$ is an integral of motion for F_μ. ■



Example: discrete time Toda lattice

Lagrange function:

$$\Lambda(x,\widetilde{x};\lambda)=\frac{1}{\lambda}\sum_{k=1}^{N}\Big(e^{\widetilde{x}_k-x_k}-1-(\widetilde{x}_k-x_k)\Big)-\lambda\sum_{k=1}^{N}e^{x_{k+1}-\widetilde{x}_k}.$$

Lagrangian equations of motion:

$$\frac{1}{\lambda^2}\Big(e^{\widetilde{x}_k-x_k}-e^{x_k-\underset{\sim}{x}_k}\Big)=e^{\underset{\sim}{x}_{k+1}-x_k}-e^{x_k-\widetilde{x}_{k-1}}.$$

Symplectic map $(x,p)\mapsto(\widetilde{x},\widetilde{p})$,

$$F_\lambda:\begin{cases} p_k=\dfrac{1}{\lambda}\Big(e^{\widetilde{x}_k-x_k}-1\Big)+\lambda e^{x_k-\widetilde{x}_{k-1}},\\[2mm] \widetilde{p}_k=\dfrac{1}{\lambda}\Big(e^{\widetilde{x}_k-x_k}-1\Big)+\lambda e^{x_{k+1}-\widetilde{x}_k}.\end{cases}$$

Integrability of discrete time Toda lattice

$\mathcal{L}$ closed on solutions $\Leftrightarrow$ generating function of integrals of motion:

$$P(x, p; \lambda) = \sum_{k=1}^{N} (\widetilde{x}_k - x_k),$$

where $\widetilde{x}_k$ should be expressed through p_k with the help of

$$e^{\widetilde{x}_k - x_k} = 1 + \lambda p_k - \lambda^2 e^{x_k - \widetilde{x}_{k-1}}.$$

One finds:

$$e^P = \mathrm{tr}\, U_N(x, p; \lambda) \cdots U_2(x, p; \lambda) U_1(x, p; \lambda),$$

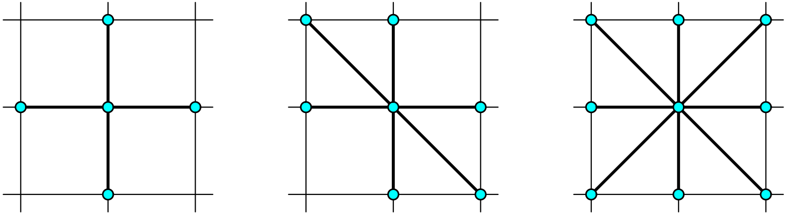
where

$$U_k(x, p; \lambda) = \begin{pmatrix} 1 + \lambda p_k & -\lambda^2 e^{x_k} \\ e^{-x_k} & 0 \end{pmatrix}.$$

Discrete space-time, $d = 2$: what is integrability of variational systems?

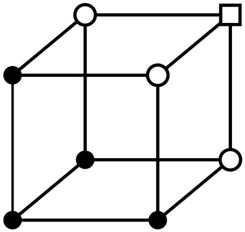
A long-standing problem: integrability of variational systems.
Solution: along the discretization path.

Equations are of elliptic type (à la Laplace equation).



Discrete hyperbolic systems: integrability $=$ multidimensional consistency

3D consistency of quad-equations $Q(x, x_i, x_j, x_{ij}) = 0$:



- Discrete analog of commuting flows
- Zero curvature representation and Bäcklund transformations
- Equation can be imposed on arbitrary quad-graphs realized as quad-surfaces in $\mathbb{Z}^m$

Difficulty: notions *evolution* and *commutativity* seem to be alien for elliptic systems. Difficult to come up with anything like 3D consistency for discrete elliptic (Laplace-type) systems.

Pluri-Lagrangian problem, discrete space-time, $d = 2$

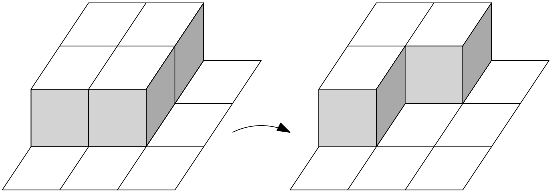
Main idea: require solutions to be extremals not only on $\mathbb{Z}^2$ but on arbitrary quad-surfaces in $\mathbb{Z}^m$.

A *discrete 2-form* $\mathcal{L}$ is a skew-symmetric function on oriented squares $\sigma_{ij} = (n, n+e_i, n+e_i+e_j, n+e_j)$ of $\mathbb{Z}^m$:

$$\mathcal{L}(\sigma_{ij}) = \Lambda_{ij}(x, x_i, x_{ij}, x_j) = -\mathcal{L}(\sigma_{ji}).$$

For a *discrete quad-surface* Σ in $\mathbb{Z}^m$, set: $S_\Sigma = \sum_{\sigma_{ij}\subset\Sigma} \mathcal{L}(\sigma_{ij})$.

Problem. Find functions $x : \mathbb{Z}^m \to \mathcal{X}$ delivering critical points for the action functional S_Σ for *any* quad-surface Σ in $\mathbb{Z}^m$.



3D corner equations

Theorem. For a given discrete 2-form $\mathcal{L}(\sigma_{ij}) = \Lambda_{ij}(x, x_i, x_{ij}, x_j)$, denote

$$S^{ijk} = d\mathcal{L}(\sigma_{ijk}) = \Delta_k \mathcal{L}(\sigma_{ij}) + \Delta_i \mathcal{L}(\sigma_{jk}) + \Delta_j \mathcal{L}(\sigma_{ki}).$$

Function $x : \mathbb{Z}^m \to \mathcal{X}$ solves the pluri-Lagrangian problem for $\mathcal{L}$, iff the following *3D corner equations* are satisfied:

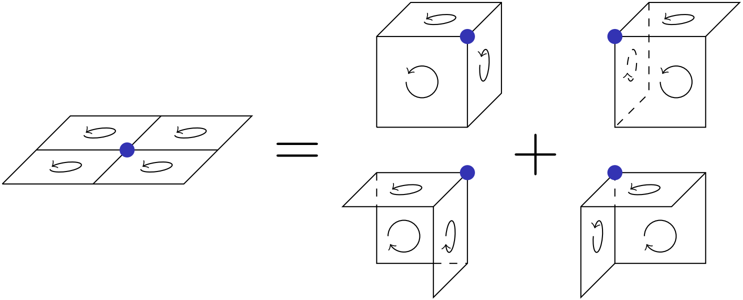
$$\frac{\partial S^{ijk}}{\partial x} = 0, \quad \frac{\partial S^{ijk}}{\partial x_i} = 0, \quad \frac{\partial S^{ijk}}{\partial x_j} = 0, \quad \frac{\partial S^{ijk}}{\partial x_k} = 0,$$

$$\frac{\partial S^{ijk}}{\partial x_{ij}} = 0, \quad \frac{\partial S^{ijk}}{\partial x_{jk}} = 0, \quad \frac{\partial S^{ijk}}{\partial x_{ik}} = 0, \quad \frac{\partial S^{ijk}}{\partial x_{ijk}} = 0$$

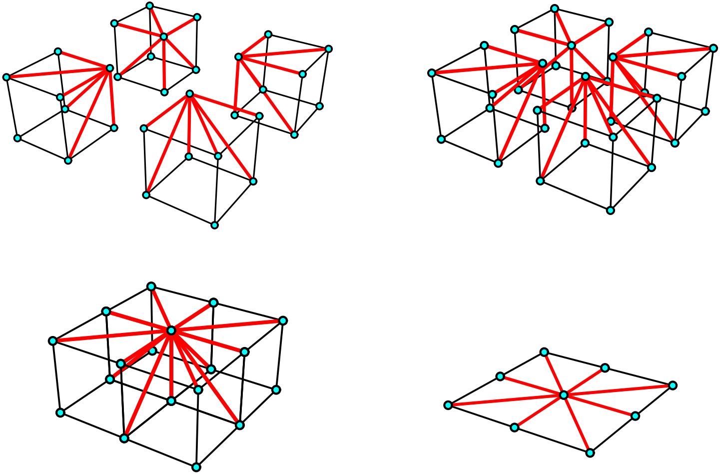
(eight equations per elementary cube σ_{ijk}). Symbolically: $\delta(d\mathcal{L}) = 0$, where δ stands for the "vertical" differential.

3D corners: elementary building blocks of quad-surfaces

Any vertex star in any quad-surface can be built from 3D corners (with extension into an extra dimension, if necessary).



3D corner equations: “elementary particles” for integrable Laplace type equations



Consistency of corner equations; almost closedness of discrete 2-form

Definition. System of corner equations is called *consistent*, if, for any cube σ_{ijk}, it has the minimal possible rank 2, i.e., if exactly two of these eight equations are independent.

Theorem. For any triple i, j, k, action S^{ijk} over an elementary cube is constant on solutions of corner equations:

$$d\mathcal{L}(\sigma_{ijk}) = S^{ijk}(x, \ldots, x_{ijk}) = c^{ijk} = \mathrm{const}$$

$$(\mathrm{mod}\ \partial S^{ijk}/\partial x = 0,\ \ldots,\ \partial S^{ijk}/\partial x_{ijk} = 0).$$

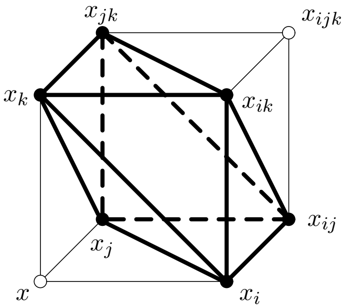
Most interesting case: all $c^{ijk} = 0$, i.e., $\mathcal{L}$ closed on solutions of corner equations.

Particular case: 3-point 2-form

For ABS equations:

$$\mathcal{L}(\sigma_{ij}) = \mathcal{L}(x, x_i, x_j; \alpha_i, \alpha_j) = L(x, x_i; \alpha_i) - L(x, x_j; \alpha_j) - \Lambda(x_i, x_j; \alpha_i, \alpha_j).$$

For such $\mathcal{L}$, action $d\mathcal{L}(\sigma_{ijk}) = S^{ijk}$ depends neither on x nor on x_{ijk}:

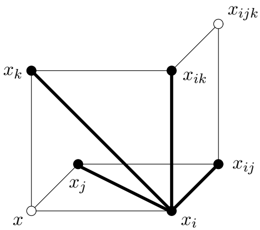


Corner equations for a 3-point 2-form

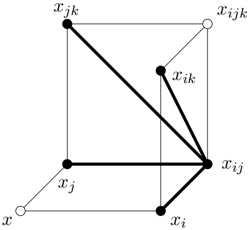
four-leg, five-point equations:

$$\psi(x_i, x_{ij}; \alpha_j) - \psi(x_i, x_{ik}; \alpha_k) - \phi(x_i, x_k; \alpha_i, \alpha_k) + \phi(x_i, x_j; \alpha_i, \alpha_j) = 0,$$

$$\psi(x_{ij}, x_i; \alpha_j) - \psi(x_{ij}, x_j; \alpha_i) - \phi(x_{ij}, x_{ik}; \alpha_j, \alpha_k) + \phi(x_{ij}, x_{jk}; \alpha_i, \alpha_k) = 0.$$



(a) 4-leg corner equation at x_i

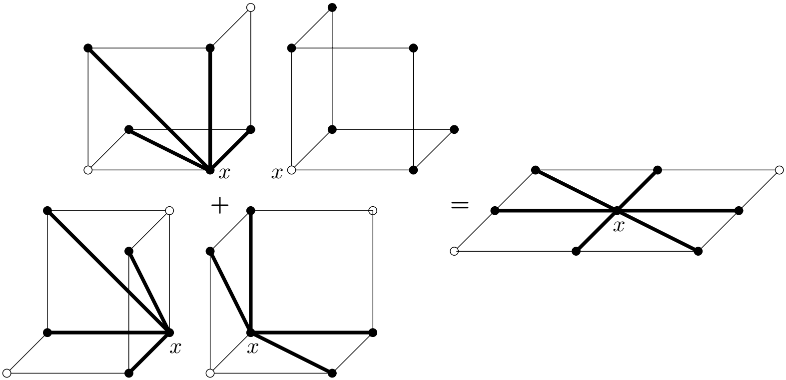


(b) 4-leg corner equation at x_{ij}

Here, we introduced the notation

$$\psi(x, y; \alpha) = \frac{\partial L(x, y; \alpha)}{\partial x}, \quad \phi(x, y; \alpha, \beta) = \frac{\partial \Lambda(x, y; \alpha, \beta)}{\partial x}.$$

From corner equations to planar Laplace type equations



Integrability of corner equations

Theorem. For the discrete 2-forms $\mathcal{L}$ for quad-equations of the ABS list, the corresponding systems of 3D corner equations are consistent, as well. Moreover, the 2-form $\mathcal{L}$ is closed on solutions of 3D corner equations.

Relation between closedness of $\mathcal{L}$ and integrability:

Theorem. System of 3D corner equations for $\mathcal{L}$ admits parameter-dependent families of *conservation laws*: $\Delta_j F_{ik} = \Delta_k F_{ij}$, where

$$F_{ij} = \frac{\partial L(x, x_i, \alpha_i)}{\partial \alpha_i} - \frac{\partial \Lambda(x_i, x_j, \alpha_i - \alpha_j)}{\partial \alpha_i}$$

($d = 2$ analog of spectrality).

Proof: Differentiate $S^{ijk} = 0$ w.r.t. α_i. $\square$

Conclusions

We propose the notion of pluri-Lagrangian systems as integrability of discrete (and continuous) variational systems.

- (Almost) closedness of the Lagrangian form ($d\mathcal{L} = \mathrm{const}$) on solutions of the pluri-Lagrangian system built-in.
- Closedness of the Lagrangian form ($d\mathcal{L} = 0$) on solutions is related to existence of integrals of motion for $d = 1$ (resp. conservation laws for $d = 2$).
- Classification of pluri-Lagrangian systems looks promising.