Geometric asymptotic reduction: the gyrokinetic model for magnetic fusion plasmas

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Outline

Tokamak physics

Gyrokinetic models

From the continuous to the discrete action

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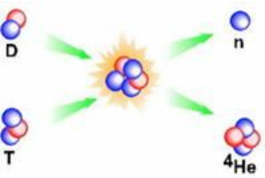
Controlled thermonuclear fusion

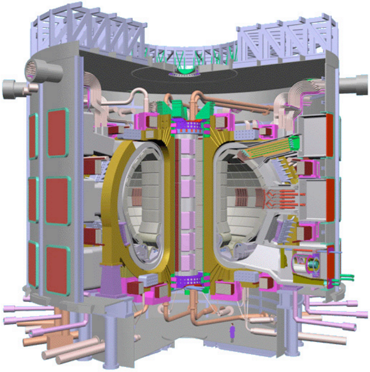


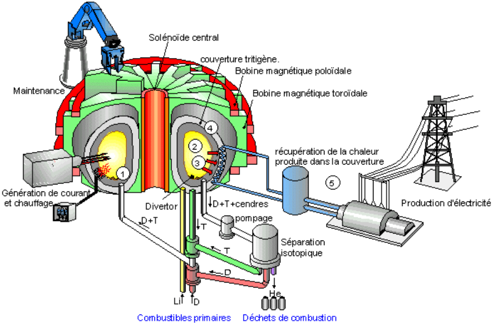
- Fusion conditions:
  $nT\tau_E$ large enough.
- $T \approx 100$ million ^oC
  fully ionized gas=plasma.

- Magnetic confinement (ITER)
- Inertial confinement
  - by laser (LMJ, NIF)
  - by heavy ions

The ITER project

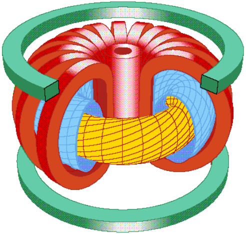
International project involving European Union, China, India, Japan, South Korea, Russia and United States aiming to prove that magnetic fusion is viable source for energy.



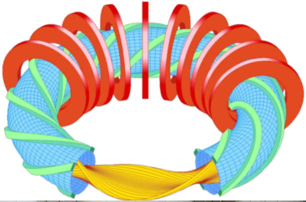


Experimental installations at IPP

Tokamak



Stellarator



Modelling of Tokamak plasmas

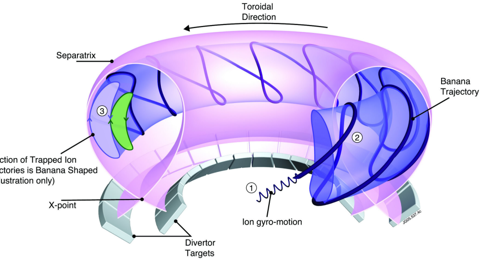
- A plasma is a collection of different species of charged particles.
- Basic model is Newton's law with pairwise interaction between particles which is largely dominated by electromagnetic force. Too many particles $n \approx 10^{19} m^{-3}$, numerically intractable.
- First reduced model: Kinetic Vlasov-Maxwell (+Landau collisions)
- Second reduced model: multi-fluid Euler-Maxwell
- Third reduced model: single fluid MHD

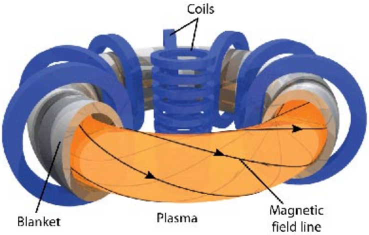
Turbulent transport in magnetized plasma

- Plasma not very collisional and far from fluid state
  $\Rightarrow$ Kinetic description necessary. Fluid and kinetic simulations of turbulent transport yield very different results.
- Vlasov (6D phase space) coupled to 3D Maxwell

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f = 0.$$

- Toroidal geometry





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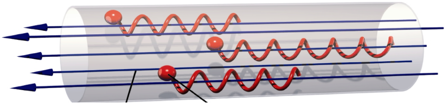
Numerical issues with 6D Vlasov-Maxwell

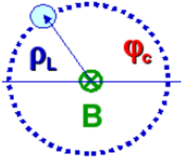
- Posed in 6D phase space! Dimension reduction if possible would help.
- Large magnetic field imposes very small time step to resolve the rotation of particles along field lines.



- Physics of interest is low frequency. Remove light waves: Darwin instead of Maxwell.
- Debye length small compared to ion Larmor radius. Quasi-neutrality assumption $n_e = n_i$ needs to be imposed instead of Poisson equation for electric field.

Towards a reduced model

- Scale separation: fast motion around magnetic field lines can be averaged out.
- Idea: separate motion of the guiding centre from rotation by a change of coordinates.



- For constant magnetic field can be done by change of coordinates: $\mathbf{X} = \mathbf{x} - \rho_L$ guiding centre + kind of cylindrical coordinates in $\mathbf{v}$: $v_\parallel$, $\mu = \frac{1}{2} m v_\perp^2 / \omega_c$, θ.
- Mixes position and velocity variables.
- Perturbative model for slowly varying magnetic field.
- Several small parameters
  - gyroperiod, Debye length
  - Magnetic field in tokamak varies slowly: $\epsilon_B = |\nabla B / B|$
  - Time dependent fluctuating fields are small.

Geometric asymptotic reduction

- Long time magnetic confinement of charged particles depends on existence of first adiabatic invariant (Northrop 1963): $\mu = \frac{1}{2} m v_{\perp}^2 / \omega_c$.
- Geometric reduction based on making this adiabatic invariant an exact invariant.
- Two steps procedure:
  - Start from Vlasov-Maxwell particle Lagrangian and reduce it using Lie transforms such that it is independent of gyromotion up to second order
  - Plug particle Lagrangian into Vlasov-Maxwell field theoretic action and perform further reduction.
- End product is gyrokinetic field theory embodied in Lagrangian. Symmetries of Lagrangian yield exact conservation laws thanks to Noether Theorem.

Historical notes

- Perturbative analysis of Vlasov:
  - linear: Rutherford & Frieman 68, Taylor & Hastie 68, Catto 78
  - non linear: Frieman & Chen 82.
- Hamiltonian methods:
  - electrostatic: Littlejohn 82, 83, Dubin 83
  - Electromagnetic: Brizard, Lee, Hahm 88, Hahm 88
- Gyrokinetic field theory:
  - Lagrangian setting: Sugama 2000, Scott & Smirnov 2010
  - Eulerian setting: Brizard 2000
- Review:
  - Brizard & Hahm 2007
  - Krommes 2012, provides a non technical review of the topic.

Motion of a particle in an electromagnetic field

- Consider given electromagnetic field defined by scalar potential ϕ and vector potential $\mathbf{A}$ such that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

- The non relativistic equations of motion of a particle in this electromagnetic field is obtained from Lagrangian (here phase space Lagrangian $\mathbf{p} \cdot \dot{\mathbf{q}} - H$ in non canonical variables for later use)

$$L_s(\mathbf{x}, \mathbf{v}, \dot{\mathbf{x}}, t) = (m_s \mathbf{v} + e_s \mathbf{A}) \cdot \dot{\mathbf{x}}^2 - (\frac{1}{2} m_s v^2 + e_s \phi).$$

where $\mathbf{p} = m_s \mathbf{v} + e_s \mathbf{A}(t, \mathbf{x})$, $H = m_s v^2/2 + e_s \phi(t, \mathbf{x})$ are canonical momentum and hamiltonian.

Abstract geometric context

- Lagrangian becomes Poincaré-Cartan 1-form
$$\gamma = \mathbf{p}\cdot \mathrm{d}\mathbf{x} - H\,\mathrm{d}t$$
with $\mathbf{p} = m_s\mathbf{v} + e_s\mathbf{A}(t,\mathbf{x})$, $H = m_s v^2/2 + e_s\phi(t,\mathbf{x})$.
- $\omega = \mathrm{d}\gamma$ is the Lagrange 2-form, which is non degenerate and so a symplectic form. Its components define the the Lagrange tensor Ω.
- Then $J = \Omega^{-1}$ is the Poisson tensor which defines the Poisson bracket
$$\{F, G\} = \nabla F^T J \nabla G$$
- The equations of motion can then be expressed from the Poisson matrix and the hamiltonian
$$\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}t} = J\nabla H.$$
- Lagrangian contains all necessary information and this structure is preserved by change of coordintates.

Derivation of gyrokinetic particle Lagrangian

- Gyrokinetic particle Lagrangian obtained from Vlasov-Maxwell particle Lagrangian by performing a change of variables, such that lowest order terms independent of gyrophase.
- This is obtained systematically order by order by the Lie transform method (Dragt & Finn 1976, Cary 1981) on the Lagrangian

$$L_s(\mathbf{x}, \mathbf{v}, \dot{\mathbf{x}}, t) = (m_s\mathbf{v} + e_s\mathbf{A}) \cdot \dot{\mathbf{x}}^2 - (\frac{1}{2}m_s|\mathbf{v}|^2 + e_s\phi).$$

- Not a unique solution.
    1. $v_\parallel$ formulation. Transform Lagrangian as is keeping fluctuation $\mathbf{A}$ in symplectic form.
    2. $p_\parallel$ formulation, $p_\parallel = v_\parallel + (e/m)A_\parallel$. Fluctuating $A_\parallel$ in hamiltonian.
    3. $u_\parallel$ formulation. Split fluctuating $A_\parallel$ into two parts. One of them goes into Hamiltonian. Includes others as special case.
- Gyrokinetic codes choose between $v_\parallel$ (symplectic) and $p_\parallel$ (hamiltonian) formulation.
- Both involve severe numerical drawbacks.

The mixed gyrokinetic particle Lagrangian

- Split $A_\parallel = A_\parallel^s + A_\parallel^h$. Define $u_\parallel = v_\parallel + (e/m)A_\parallel^h$
- The gyrokinetic Lagrangian for a single particle always in the form

$$L = \mathbf{A}^* \cdot \dot{\mathbf{X}} + \mu\dot{\theta} - H$$

$$\text{with } \mathbf{A}^* = \mathbf{A}_0 + \Big((m_s/e_s)u_\parallel + \langle A_\parallel^s \rangle\Big)\,\mathbf{b}, \quad \mathbf{b} = \mathbf{B}/B,$$

$$H = H_0 + H_1 + H_2, \quad H_0 = \frac{1}{2}m_s u_\parallel^2 + \mu B, \quad H_1 = \langle \phi - u_\parallel A_\parallel^h \rangle$$

where

$$\langle \psi \rangle(\mathbf{x}, \mu) \stackrel{\text{def}}{=} \frac{1}{2\pi} \oint \psi(\mathbf{x} + \rho)\,\mathrm{d}\alpha.$$

- Perpendicular component of fluctuating vector potential **A** neglected.

The Vlasov equation

- Consider a population of particles evolving with

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{V}, \quad \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{F} = \frac{e}{m}(\mathbf{E} + \mathbf{V} \times \mathbf{B}).$$

- Their distribution function f, or more precisely probability density in phase space (up to normalisation), satisfies the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \mathbf{F} \cdot \nabla_v f = 0.$$

- Given an initial distribution f_0, the distribution at time t is equivalently characterised by the solution of the Vlasov equation or the particle positions $f(t, \mathbf{z}) = f_0(X(0; \mathbf{z}, t), V(0; \mathbf{z}, t))$, where we denote by $\mathbf{z} = (\mathbf{x}, \mathbf{v})$.

Action principle for the Vlasov-Maxwell equations

- Field theory is action principle from which Vlasov-Maxwell equations are derived.
- Action proposed by Low (1958) with a Lagrangian formulation for Vlasov, *i.e.* based on characteristics.
- Based on particle Lagrangian for species s, L_s.
- Such an action, splitting between particle and field Lagrangian, using standard non canonical coordinates, reads:

$$\mathcal{S} = \sum_{s} \int f_s(\mathbf{z}_0, t_0) L_s(\mathbf{X}(\mathbf{z}_0, t_0; t), \dot{\mathbf{X}}(\mathbf{z}_0, t_0; t), t) \, \mathrm{d}\mathbf{z}_0 \, \mathrm{d}t$$
$$+ \frac{\epsilon_0}{2} \int |\nabla \phi + \frac{\partial \mathbf{A}}{\partial t}|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t - \frac{1}{2\mu_0} \int |\nabla \times \mathbf{A}|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t.$$

Particle distribution functions f_s taken at initial time.

The electromagnetic gyrokinetic field theory

- Gyrokinetics is a low frequency approximation.
  Darwin approximation: $\partial_t \mathbf{A}$ removed from Lagrangian.
- Quasi-neutrality approximation: $|\nabla\phi|^2$ removed:

$$\mathcal{S} = \sum_{\mathrm{s}} \int f_s(\mathbf{z}_0, t_0)(\mathbf{A}^* \cdot \dot{\mathbf{X}} - H)\,\mathrm{d}\mathbf{z}_0 - \frac{1}{2\mu_0}\int |\nabla \times (A_{\|}\mathbf{b})|^2\,\mathrm{d}\mathbf{x}.$$

- Additional approximation made to avoid fully implicit formulation:
  Second order term in Lagrangian linearised (consistent with ordering) by replacing full f by background f_M

$$\mathcal{S} = \sum_{\mathrm{s}} \int f_s(\mathbf{z}_0, t_0)(\mathbf{A}^* \cdot \dot{\mathbf{X}} - H_0 - H_1)\,\mathrm{d}\mathbf{z}_0$$
$$- \sum_{\mathrm{s}} \int f_{M,s}(\mathbf{z}_0) H_2\,\mathrm{d}\mathbf{z}_0 - \frac{1}{2\mu_0}\int |\nabla \times (A_{\|}\mathbf{b})|^2\,\mathrm{d}\mathbf{x}.$$

Derivation of the gyrokinetic equations from the action principle

We denote by $\mathbf{B}^* = \nabla \times \mathbf{A}^*$ and $B_\parallel^* = \mathbf{B}^* \cdot \mathbf{b}$.

- Setting $\frac{\delta \mathcal{S}}{\delta Z_i} = 0$, $i = 1, 2, 3, 4$ yields:

$$\mathbf{B}^* \times \dot{\mathbf{R}} = -\frac{m}{q}\dot{P}_\parallel \mathbf{b} - \frac{1}{q}\nabla(H_0 + H_1), \quad \mathbf{b} \cdot \dot{\mathbf{R}} = \frac{1}{m}\frac{\partial(H_0 + H_1)}{\partial p_\parallel}.$$

- Solving for $\dot{\mathbf{R}}$ and $\dot{P}_\parallel$ we get the equations of motion of the gyrocenters:

$$B_\parallel^* \dot{\mathbf{R}} = \frac{1}{m}\frac{\partial(H_0 + H_1)}{\partial p_\parallel}\mathbf{B}^* - \frac{1}{q}\nabla(H_0 + H_1) \times \mathbf{b}, \; B_\parallel^* \dot{P}_\parallel = -\frac{1}{m}\nabla(H_0 + H_1) \cdot \mathbf{B}^*.$$

- These are the characteristics of the gyrokinetic Vlasov equation

$$\frac{\partial f}{\partial t} + \dot{\mathbf{R}} \cdot \nabla f + \dot{P}_\parallel \frac{\partial f}{\partial p_\parallel} = 0.$$

Gyrokinetic Ampere and Poisson equations

- The gyrokinetic Poisson (or rather quasi-neutrality) equation is obtained by variations with respect to ϕ

$$\int \frac{e_i^2 \rho_i^2 n_{s,0}}{k_\mathrm{B} T_i} \nabla_\perp \phi \cdot \nabla \tilde{\phi} \,\mathrm{d}\mathbf{x} = \int qn \langle \tilde{\phi} \rangle \,\mathrm{d}\mathbf{x}, \quad \forall \tilde{\phi}$$

- The gyrokinetic Ampère equation is obtained by variations with respect to $A_\|$:

$$\int \nabla_\perp A_\| \cdot \nabla_\perp \tilde{A}_\|^h \,\mathrm{d}\mathbf{x} + \sum_s \int \frac{\mu_0 q_s^2 n_s}{m_s} \langle A_\|^h \rangle \langle \tilde{A}_\|^h \rangle \,\mathrm{d}\mathbf{x}$$
$$= \mu_0 \int j_\| \langle \tilde{A}_\|^h \rangle \,\mathrm{d}\mathbf{x}, \quad \forall \tilde{A}_\|^h$$

- where $A_\| = A_\|^s + A_\|^h$ and $A_\|^s$ is related to ϕ by the constraint

$$\frac{\partial A_\|^s}{\partial t} + \nabla \phi \cdot \mathbf{b} = 0.$$

Conserved quantities

- Symmetries of Lagrangian yield invariants using Noether's theorem
- Time translation: Conservation of energy:

$$\mathcal{E}(t) = \sum_s \int \mathrm{d}W_0 \mathrm{d}V_0 f_{s,0}(\mathbf{z}_0) H_s - \int \mathrm{d}V \frac{e_i^2 \rho_i^2 n_{s,0}}{k_\mathrm{B} T_i} |\nabla \phi|^2 + \frac{1}{2\mu_0} \int \mathrm{d}V |\nabla_\perp A_\|}|^2.$$

- Axisymmetry of background vector potential:
  Conservation of total canonical angular momentum:

$$\mathcal{P}_\varphi = \sum_s e_s \int \mathrm{d}W_0 \mathrm{d}V_0 f_{s,0}(\mathbf{z}_0) \mathbf{A}^\star_{s,\varphi}$$

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Discretisation of the action

- Our action principles rely on a Lagrangian (as opposed to Eulerian) formulation of the Vlasov equation: the functionals on which our action depends are the characteristics of the Vlasov equations $\mathbf{X}$ and $\mathbf{V}$ in addition to the scalar and vector potentials ϕ and $\mathbf{A}$.
- A natural discretisation relies on:
  - A Monte-Carlo discretisation of the phase space at the initial time: select randomly some initial positions of the particles.
  - Approximate the continuous function spaces for ϕ and $\mathbf{A}$ by discrete subspaces.
  - Yields a discrete action where a finite (large) number of scalars are varied: the particle phase space positions and coefficients in Finite Element basis.
- When performing the variations, we get the classical Particle In Cell Finite Element Method (PIC-FEM).

FEEC needed for Maxwell's equations

- In order to preserve the continuous structure at the discrete level, the different unknowns ϕ, **A**, **E** and **B** need to be chosen in compatible Finite Element spaces.
- This is provided by Finite Element Exterior Calculus (FEEC) introduced by Arnold, Falk and Winther.
- Continuous and discrete complexes are the following

$$
\begin{array}{ccccccc}
 & \textbf{grad} & & \textbf{curl} & & \mathrm{div} & \\
H^1(\Omega) & \longrightarrow & H(\textbf{curl},\Omega) & \longrightarrow & H(\mathrm{div},\Omega) & \longrightarrow & L^2(\Omega) \\
\downarrow \Pi_0 & & \downarrow \Pi_1 & & \downarrow \Pi_2 & & \downarrow \Pi_3 \\
V_0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3
\end{array}
$$

- Faraday and $\mathrm{div}\, B = 0$ verified strongly as

$$
{}^1\mathbf{E} = -\nabla\, {}^0\phi - \frac{\partial\, {}^1\mathbf{A}}{\partial t}, \qquad {}^2\mathbf{B} = \nabla \times\, {}^1\mathbf{A}.
$$

- Ampere and Gauss' law obtained from variations of FE coefficients.

PIC Finite Element approximation of the Action

- Compatible FE discretisation:

$$\phi_h \in V_0, \quad \mathbf{A}_h, \mathbf{E}_h \in V_1, \mathbf{B}_h \in V_2.$$

- Particle discretisation of $f \approx \sum_k w_k \delta(x - x_k(t))\delta(v - v_k(t))$
- Vlasov-Maxwell action becomes:

$$\mathcal{S}_{N,h} = \sum_{k=1}^{N} w_k L_s(\mathbf{Z}(\mathbf{z}_{k,0}, t_0; t), \dot{\mathbf{Z}}(\mathbf{z}_{k,0}, t_0; t), t) - \frac{1}{2}\int |\sum_{i=1}^{N_g} a_i(t)\nabla\times\Lambda_i^1(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}$$
$$+\frac{1}{2}\int \left|\sum_{i=1}^{N_g}\phi_i(t)\nabla\Lambda_i^0(\mathbf{x}) + \sum_{i=1}^{N_g}\frac{\mathrm{d}a_i(t)}{\mathrm{d}t}\Lambda_i^1(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}.$$

- $\mathbf{Z}(\mathbf{z}_{k,0}, t_0; t)$ will be traditionally denoted by $\mathbf{z}_k(t)$ is the phase space position at time t of the particle that was at $\mathbf{z}_{k,0}$ at time t_0.

PIC-FE discretisation of the action

- We know have a discrete action depending on particle positions and Finite Element degrees of freedom, which define the generalised coordinates

$$\mathcal{S}_{N,h}[\mathbf{x}_1, \dots, \mathbf{x}_N, \dot{\mathbf{x}}_1, \dots, \dot{\mathbf{x}}_N, \mathbf{v}_1, \dots, \mathbf{v}_N, \phi_1, \dots, \phi_{N_g}, a_1, \dots, a_{N_g}]$$

- The discrete electric and magnetic fields are defined exactly as in the continuous case from the discrete potentials thanks to the compatible Finite Element spaces

$$\mathbf{E}_h = \sum_i e_i \Lambda_i^1(\mathbf{x}) = -\nabla\phi_h - \frac{\partial \mathbf{A}_h}{\partial t}, \quad \mathbf{B}_h = \sum b_i \Lambda_i^2(\mathbf{x}) = \nabla \times \mathbf{A}_h.$$

- It immediately follows like in the continuous case the discrete Faraday equation

$$\frac{\partial \mathbf{B}_h}{\partial t} + \nabla \times \mathbf{E}_h = 0.$$

Time advance via Hamiltonian splitting

- Following the prescription of Crouseilles-Einkemmer-Faou a Hamiltonian splitting can be performed, treating the three terms of the Hamiltonian separately

$$H = \frac{1}{2}\mathbf{v}M_p\mathbf{v} + \frac{1}{2}\mathbf{e}M_1\mathbf{e} + \frac{1}{2}\mathbf{b}M_2\mathbf{b} = H_p + H_e + H_b.$$

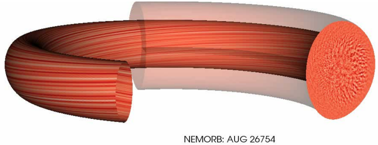
- Split and solve successively ($\Omega(u)$ Poisson matrix)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \Omega(u)\nabla H_i, \quad i = p, e, b$$

- Lie-Trotter splitting (first order), Strang splitting (second order) or even higher order.
- Exact solution possible for H_e and H_b.
- For H_p split further between the three components. Other possibility: use variational integrator

Comments and related work

- Variational FE-PIC codes along with control variates for noise reduction at the base of success of PIC simulations of Tokamak turbulence with ORB5 family of codes.



(Picture: A. Bottino)

- A lot of recent effort towards variational or Hamiltonian discretisation of Vlasov (or related)
  - First ref: Lewis, Energy conserving numerical approximations of Vlasov plasmas, JCP 1970
  - Shadwick, Stamm, Estatiev, Variational formulation of macro-particle plasma simulation algorithms (Phys Plasmas 2014)
  - Squire, Qin, Tang, Geometric integration of the Vlasov-Maxwell system with a variational particle-in-cell scheme, (Phys Plasmas 2012)