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Hans Munthe-Kaas

Lie-Butcher series and differential geometry. Klein geometries and symmetric spaces

Hans Munthe-Kaas,
Univ. of Bergen, Norway
http://hans.munthe-kaas.no
hans.munthe-kaas@uib.no

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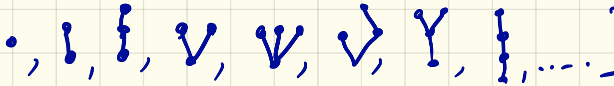
Butcher B-series

(E. Hairer, G. Wanner 1974)

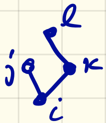
Important view:
B-series are Taylor expansion of the map
$f \mapsto B_f(\alpha)$
with values in space of differential operators

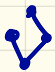
$$\boxed{B_f(\alpha, t, y) := y + \sum_{\tau \in T} \frac{t^{|\tau|}\alpha(\tau)}{\sigma(\tau)} \mathcal{F}_f(\tau)(y)}$$

$T = \{$  $, \dots \}$

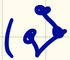
$\alpha : T \to \mathbb{R}$

$\mathcal{F}_f(\tau)$: Elementary differentials



Ex: $\mathcal{F}_f($  $) = f''(f, f'(f)) = \sum_{jkl} f^i_{jk} f^j f^k_l f^l = \sum_{jkl} f^j f^l \frac{\partial^2 f^i}{\partial x_j \partial x_k} \frac{\partial f^k}{\partial x_l}$

$\sigma(\tau)$: symmetry factor.

$\sigma($  $) = 2$, $\sigma($  $) = 6$, $\sigma($  $) = 1$

③ Differential geometry, connections and parallel transport.

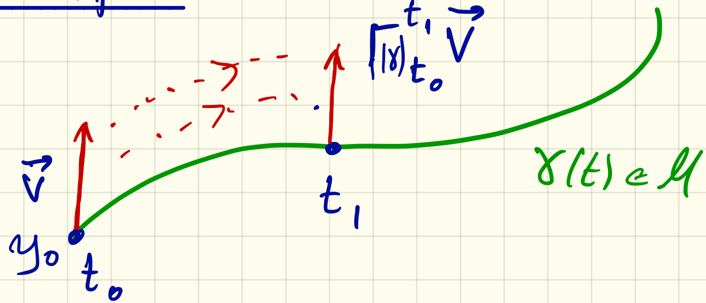
$$\boxed{\underline{\text{Connection}}: \quad \nabla_f g \equiv f \triangleright g \quad : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)}$$

$\underline{\text{DEF}}$: $(\varphi f) \triangleright g = \varphi (f \triangleright g)$, $\quad f \triangleright (\varphi g) = f(\varphi) g + \varphi (f \triangleright g)$, $\quad \varphi \in \mathcal{F}(M)$

$\underline{\text{Ex}}$: Standard connection on $\mathbb{R}^n$:

$$\boxed{(f \triangleright g)^j = \sum_{i=1}^{n} f^i \frac{\partial g^j}{\partial x^i}}$$

Parallel transport:



$$\dot{\gamma} = f(\gamma(t))$$
$$\gamma(0) = y_0$$

Parallel transport pullback:

$$\Psi^*_{t,f}(g)(y_0) := \Gamma(\gamma)^0_t \, g(\gamma(t))$$

Par. transp $\Rightarrow$ connection

$$\boxed{f \triangleright g = \frac{\partial}{\partial t}\Big|_{t=0} \Psi^*_{t,f}\, g}$$

(5) Connection $\Rightarrow$ parallel transport

$$\boxed{\frac{\partial}{\partial t}\Psi_{t,f}^{*}g = \Psi_{t,f}^{*}\, f \triangleright g}$$

$$\Rightarrow \frac{\partial^n}{\partial t^n}\Big|_{t=0}\Psi_{t,f}^{*}g = f \triangleright (f \triangleright (f \triangleright \ldots \triangleright (f \triangleright g)))$$

$$\Rightarrow \boxed{\Psi_{t,f}^{*}(g) = g + t f \triangleright g + \frac{t^2}{2} f \triangleright (f \triangleright g) + \frac{t^3}{3!} f \triangleright (f \triangleright (f \triangleright g)) + \ldots}$$

If we can extend $\triangleright$ to higher order diff. operators such that $f \triangleright (f \triangleright f) = (f * f) \triangleright f$ then

$$\boxed{\begin{aligned}\Psi_{t,f}^{*}g &= g + t f \triangleright g + \frac{t^2}{2}(f * f) \triangleright g + \ldots = \exp^{*}(tf) \triangleright g \\ \exp^{*}(tf) &= \mathbb{1} + tf + \frac{t^2}{2} f * f + \frac{t^3}{3!} f * f * f + \ldots\end{aligned}}$$

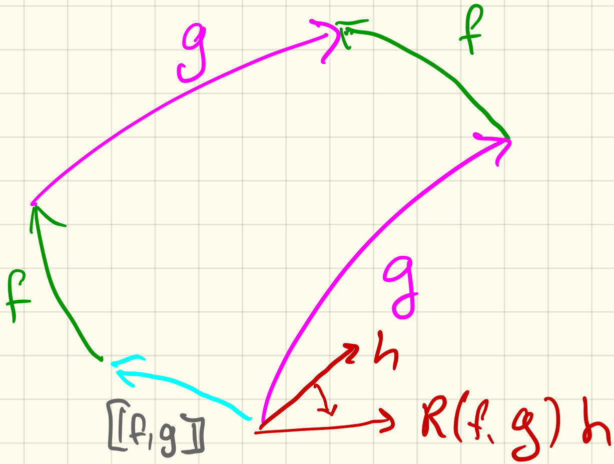
6 Curvature tensor: $R : \mathfrak{X}(M) \wedge \mathfrak{X}(M) \to \mathrm{End}(M)$

Jacobi bracket.

If $\quad f \triangleright (g \triangleright h) = (f * g) \triangleright h \quad$ (extension of $\triangleright$)

then $\quad f \triangleright (g \triangleright h) - g \triangleright (f \triangleright h) = (f * g - g * f) \triangleright h = [\![f, g]\!] \triangleright h.$

$$\boxed{R(f,g)h := f \triangleright (g \triangleright h) - g \triangleright (f \triangleright h) - [\![f, g]\!] \triangleright h}$$



Extension of $\triangleright$ to $\mathcal{U}(\mathfrak{X}(M))$ is only possible if

$$R = 0$$

(flat connection)

7 Curvature and torsion

$$\boxed{\begin{aligned} R(f,g)h &:= f \triangleright (g \triangleright h) - g \triangleright (f \triangleright h) - [\![f,g]\!] \triangleright h \\ T(f,g) &:= f \triangleright g - g \triangleright f - [\![f,g]\!] =: -[f,g] \end{aligned}}$$

$$R(f,g)h - T(f,g) \triangleright h = \underbrace{a(f,g,h) - a(g,f,h)}_{} =: \langle f,g,h \rangle$$

triple bracket

$$a(f,g,h) = f \triangleright (g \triangleright h) - (f \triangleright g) \triangleright h$$

associator

8 Nomizu's classification of invariant connections (1954)

	$T=0$	$\nabla T=0$
$R=0$	Euclidean Space PreLie algebra	Lie groups (and Klein geometries) PostLie alg.
$\nabla R=0$	Symmetric space PostLieTriple algebra	Reductive homogeneous space

PreLie:

$$\langle f, g, h\rangle = 0.$$

PostLie:

$[-,-] = -T(x,y)$ Lie

$$\langle f,g,h\rangle = [f,g] \triangleright h$$

$$f \triangleright [g,h] = [f \triangleright g, h] + [f, g \triangleright h]$$

9 Series expansions in $\triangleright$ and RK-methods

Case $R=0$ (pre/post Lie)

$$\boxed{\Psi^*_{t,f}\, g = \exp^*(tf) \triangleright g}$$

Two associative products: $\boxed{f * g \,,\quad f \cdot g := f*g - f \triangleright g}$

Two flowmaps: $\exp^*(tf)$: Exact solution

$\exp^\cdot(tf)$: Geodesic. $\equiv$ Eulers method.

Fundamental problem: Approximate $\exp^*$ with $\exp^\cdot$

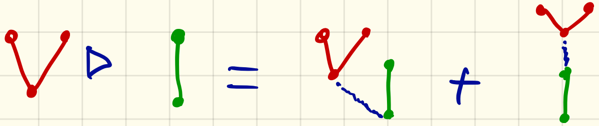
RK:

$$\boxed{\begin{aligned} K_i &= \exp^\cdot\Big(\sum_j a_{ij} K_j\Big) \triangleright f \,, \quad i = 1 \dots s \\ \Psi_{RK}(f) &:= \exp^\cdot\Big(\sum_j b_j K_j\Big) \end{aligned}}$$

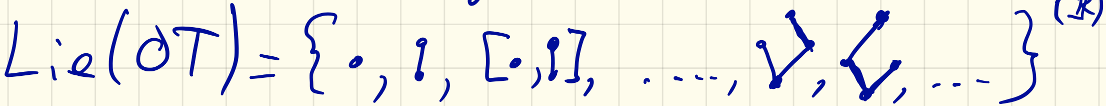
$f \mapsto \Psi_{RK}(f)$ has an expansion in $\triangleright$ (B- or LB-series)

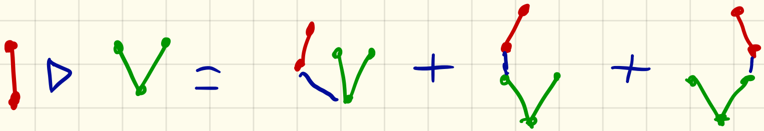
10 Free algebras ("Generic" algebras).

Free PreLie: Trees with grafting



Free PostLie: Lie algebra of ordered trees w/left grafting





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Lie triple systems: (LTS)

$[x,y]$: Lie $\Rightarrow$ $[x,y,z] := [[x,y],z]$ is LTS.

Definition: LTS : $(A, [x,y,z])$ where:

(i) $[x,y,z]$ is trilinear on A

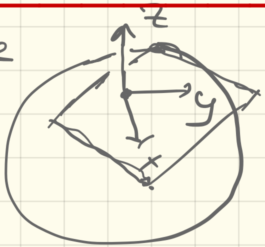
(ii) $[x,y,z] = [y,x,z]$

(iii) $[x,y,z] + [y,z,x] + [z,x,y] = 0$

(iv) $[x,y,[z,u,v]] = [[x,y,z],u,v] + [z,[x,y,u]] + [z,u,[x,y,v]]$

LTS describes TM for M symmetric space.

Example: Sphere



S^2

$[x,y] = x \times y$

(12) PostLie Triple algebra (MK-Føllesdal).

Definition: pltr $(A, \rhd)$ is pltr if

$$\langle x,y,z\rangle := a(x,y,z) - a(y,x,z)$$

is a LTS, and

$$x \rhd \langle y,z,t\rangle = \langle x\rhd y, z, t\rangle + \langle y, x\rhd z, t\rangle + \langle y, z, x\rhd t\rangle$$

! The canonical connection on a symmetric space is a plts !

(But does this definition capture all essential info?)

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<u>Theorem:</u>

$$\boxed{\begin{array}{l}\text{Free pltr algebra} = LTS(T) \\ (\text{Free Lie triple system over (un-ordered) trees})\end{array}}$$

<u>Construction:</u>

$$\boxed{LTS(T) = \text{odd commutators in } Lie(T)}$$

Counting graded components? (Formula not known)

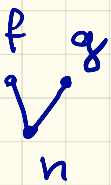
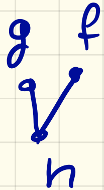
13b

Why is free PostLieTriple = LTS(T)?

$$\langle f,g,h\rangle = a(f,g,h) - a(g,f,h)$$

Consider left grafting on <u>ordered</u> trees:

$$a(f,g,h) = f\triangleright(g\triangleright h) - (f\triangleright g)\triangleright h =$$ 

$$\langle f,g,h\rangle =$$  $-$ 

We can re-order branches at the cost of producing triple brackets. By rewriting we identify quotient by LTS(T)

(14) Klein geometries and post Lie algebras

$H < G$ (closed Lie subgroup)

$$\underset{\text{fibre (isotropy)}}{H} \to \underset{\text{principal } H\text{-bundle}}{G} \to \underset{\text{homogeneous space}}{M} = G/H, \qquad M = \{gH\}$$

Left Maurer-Cartan form : $\eta : TG \to \mathfrak{g}$, $gV \mapsto V$

(i) $\eta(\xi_v) = V$ for all $v \in \mathfrak{h}$ (Lie alg. of H)

(ii) $R_h^* \eta = \mathrm{Ad}_{h^{-1}} \eta$ for all $h \in H$

(iii) $d\eta + \frac{1}{2}[\eta, \eta] = 0$ (flat cartan connection)

15 Homogeneous manifolds have two different postLie algebras

Standard formulation LGI on $\mathcal{M} = G/H$:

$$f : \mathcal{M} \to \mathfrak{g} \implies F(y) = f(y)\cdot y, \quad F \in \mathcal{X}(\mathcal{M})$$

$$f \triangleright g = F(g) \quad \text{i.e.} \left((f \triangleright g)(p) = \left.\frac{\partial}{\partial t}\right|_{t=0} g(\exp(t f(p))\cdot p) \right)$$

$$[f, g](p) = [f(p), g(p)]_{\mathfrak{g}}$$

postLie.

The standard formulation is bad geometrically, especially for symmetric spaces.

Alternative formulation: (Cartan view)

$$f : G \longrightarrow \mathfrak{g} \quad \text{s.t.} \quad f(gh) = \mathrm{Ad}_{h^{-1}} f(g)$$

$$f \Longleftrightarrow F \in \mathcal{X}(M),$$

where

$$F(gH) = \frac{\partial}{\partial t}\Big|_{t=0} gh \exp(t f(gh)) H$$

$$f \triangleright \tilde{f} := F(\tilde{f})$$

$$[f, \tilde{f}] := -[f, \tilde{f}]_{\mathfrak{g}}$$

post Lie

17 Symmetric space

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$$

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$$

$$[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$$

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$$

$$\begin{tikzcd}
TG \arrow[r, "m"] \arrow[d] & \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \\
G \arrow[ur, "f"] \arrow[d, "j"] & TM \\
iM \arrow[ur, "F"'] &
\end{tikzcd}$$

Advantage of alternative formulation:

$f : G \to \mathfrak{g}$ splits $f = f_+ + f_-$

$f_+ : G \to \mathfrak{h}$, $f_- : G \to \mathfrak{k}$

Invariant under parallel transport.

17b

Thm: Let $(P, \rhd, [\cdot,\cdot])$ be post Lie and $P = P_- \oplus P_+$ a splitting s.t.
$[P_+, P_+] \subset P_+$, $[P_+, P_-] \subset P_-$, $[P_-, P_-] \subset P_+$
and furthermore

$$P_- \rhd P_- \subseteq P_-$$

$$f_+ \rhd f_- = [f_+, f_-]$$

Then $(P, \rhd)$ is postLieTriple

17c

Thm. If $\{P, \triangleright, [\cdot,\cdot]\}$ is postLie then

$f \blacktriangleright g = f \triangleright g + [f,g]$ is also post Lie and

$f \succ g = f \triangleright g + \frac{1}{2}$ is postLie Triple

(cfr. 0, +, − Cartan-Schouten brackets)

(18) Concluding remarks

- We have new geometric integrators on symmetric spaces based on parallel transport in canonical connection ("Levi-Civita")
- pre Lie $\Rightarrow$ Butcher group w/ parallel transport.
- post Lie $\Rightarrow$ Lie Butcher group w/ parallel transport.
- post Lie + involutive automorphism $\Rightarrow$ plts
- plts $\Rightarrow$ post Lie + invol. aut.

  This shows that plts captures all essential info!

- Software package is being written (Haskell).
  postLie, preLie, postLieTriple