Differential invariant signatures (after Olver) for images

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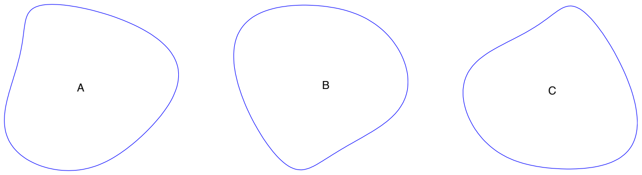
Comparing planar objects

"Shape is the ultimate nonlinear thing" – David Mumford, ICM, 2002.



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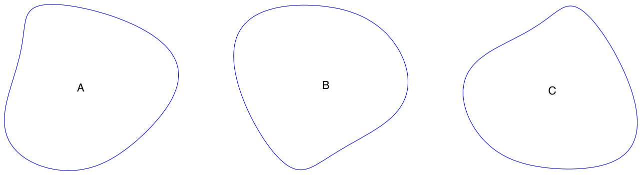

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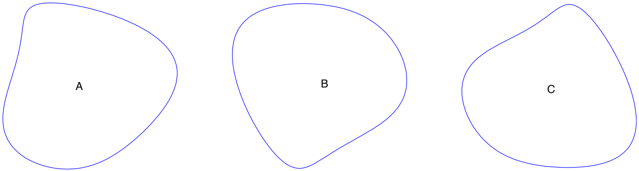

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Length(A) = Length(C) = 6.4934; Length(B) = 6.4377.

A and C are identical modulo rotations.

B is about 1% different in the L^2 norm.

Comparing planar objects

Given a set of planar objects, we may want to compare them modulo a transformation group G.

Two main approaches:

1. *Registration*: For each pair a, b of objects,

$$\min_{g \in G} \|g \cdot a - b\|$$

2. *Invariants*: Use a G-invariant representation of the objects: a and b have the same internal representation iff $b = g \cdot a$

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2. *Invariants*: Use a G-invariant representation of the objects: a and b have the same internal representation iff $b = g \cdot a$
   - *Partial invariants:* Use I where $I(a) = I(g \cdot a)$ for all $g \in G$. Example: length of a curve under Euclidean group.

The setting

1. The *objects* may be of various kinds:
   1. Shapes: unparameterized planar curves
   2. Images: $f : [0, 1]^2 \to \mathbb{R}^k$, $k =$ number of color channels in the image

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    1. $E(2)$, $SE(2)$, $Sim(2)$ (Euclidean groups)
    2. $A(2)$ and $SA(2)$ (affine groups)
    3. $PSL(3, \mathbb{R})$ (projective group)
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3. Applications:
    1. object recognition
    2. pattern matching
    3. feature detection, tracking, shape analysis, tomography, ...

Examples of Möbius shapes in nature

From S V Petukhov's *Non-Euclidean geometries and algorithms of living bodies*, 1989:

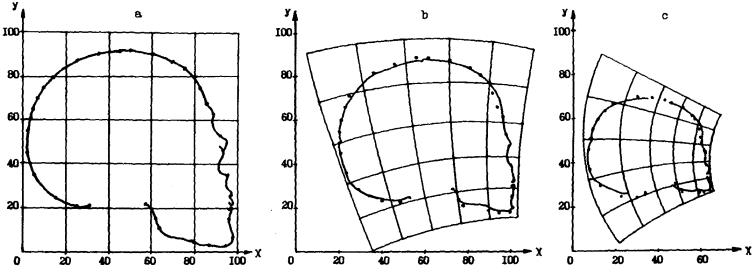

Fig. 22. Möbius transformations in the modeling of ontogenetic transformations of the human skull. Profiles of the skulls of an adult (a), a 5-year-old (b) and a newborn (c), taken from Ref. [32].

Growth of a human skull

The talk

1. Part I: Differential signatures of images
   (Stephen Marsland, Richard Brown)

2. Part II: Currents and finite elements as a tool for shape space
   (Marsland, Klas Modin, Olivier Verdier)

Invariants in mathematics

There is a vast literature on invariants in mathematics. . .

1. There is an algorithm (the moving frame method) to construct a minimal set of invariants for specific group actions.
2. There is classical invariant theory, which e.g seeks a *First Fundamental Theorem* for each group action, i.e., the set of all invariants of a given type (e.g. polynomial).

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3. Hardly any such FFTs are known.
4. An FFT does not guarantee that the invariants distinguish the group orbits.
5. The definition of an invariant only says $I(x) = I(g \cdot x)$.
   It says nothing about $I(x) - I(y)$ when x and y are in different orbits.

Invariants in computer science

There is vast literature on invariants in computer science.

An invariant should offer:

1. fast computation
2. good discrimination (A, B far apart iff their invariants are far apart)
3. completeness (A, B have same invariants iff they are the same)
4. stability (invariants nearby implies A, B nearby)
5. robustness (if B is a noisy A, their invariants should be nearby)

Peter Olver's main example

$G = SE(2)$ acts on curves $\phi\colon S^1 \to \mathbb{R}^2$ by $g \cdot \phi = g \circ \phi$.

The set

$$\{(\kappa(t), \kappa_s(t)\colon t \in [0,1)\} \subset \mathbb{R}^2$$

is a differential invariant signature for Euclidean curves.

It is also invariant under parameterizations, i.e.

$$\psi \cdot \phi := \phi \circ \psi, \quad \psi \in \mathrm{Diff}(S^1).$$

Research goal:

Systematically construct differential invariant signatures for k-colour planar images

$$f\colon \mathbb{R}^2 \to \mathbb{R}^k$$

with respect to the action of a planar group G, where

$$g \cdot f := f \circ g^{-1}.$$

Example: $G = E(2)$

For 1-colour images $f\colon \mathbb{R}^2 \to \mathbb{R}$, the set

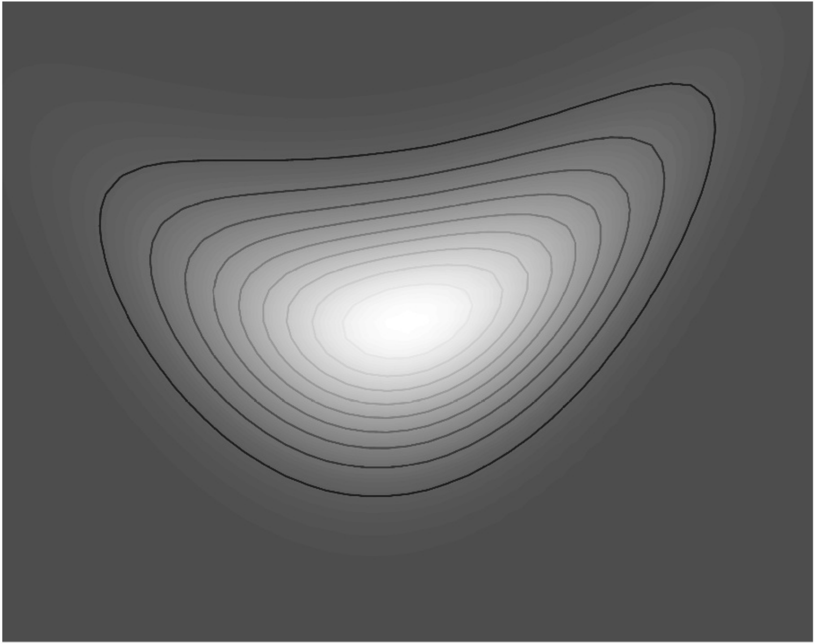
$$\left\{ \left(f,\, \|\nabla f\|^2,\, \nabla^2 f\right)(x,y) \colon (x,y) \in \mathbb{R}^2 \right\}$$

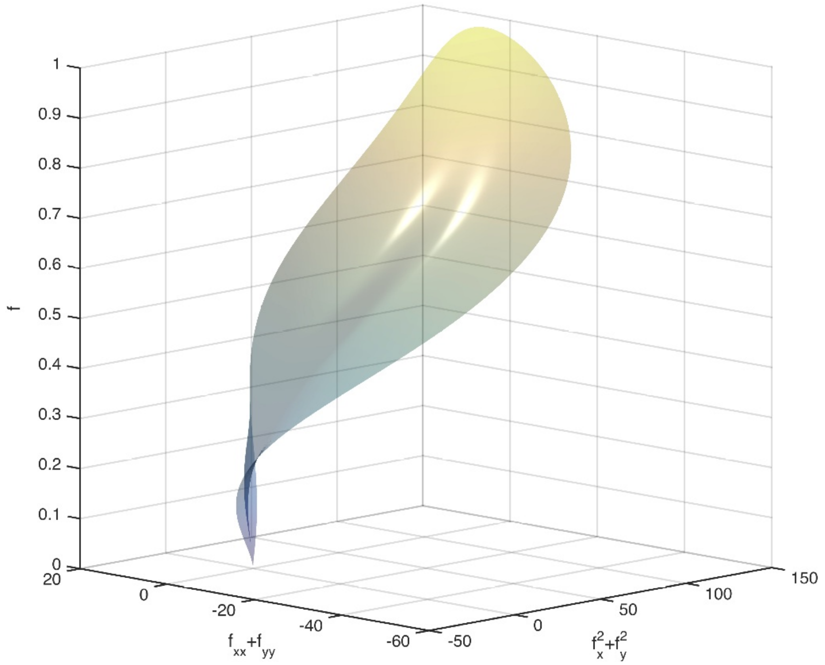
is a differential invariant.

It is an immersed 2-dimensional submanifold of $\mathbb{R}^3$.

Where does this come from?

$f = \exp(-4\,x^2 - 8(y - 0.2\,x - 0.8\,x^2)^2)$





f
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
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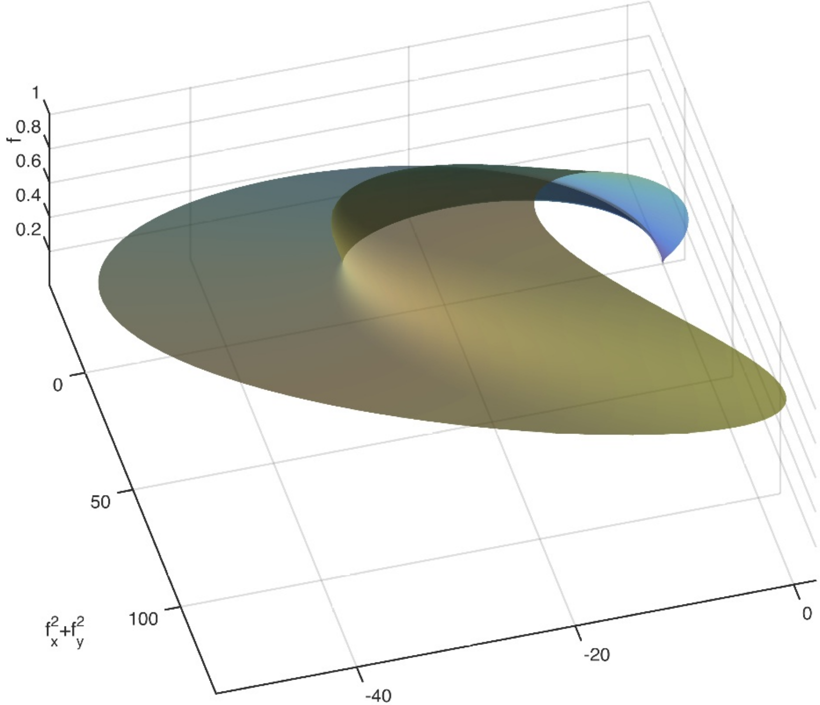
0
20
0
-20
-40
-60
$f_{xx}+f_{yy}$
-50
0
50
100
150
$f_x^2+f_y^2$



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Example: $G = E(2)$

1. Moving frame method:
   - First prolong the group action $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ to get

$$f \mapsto f, \quad f_i \mapsto A_{ij} f_j, \quad f_{ij} \mapsto A_{ij} A_{kl} f_{jl}, \ldots .$$

   - Then step-by-step solve for the group parameters to put $(f, f_i, \ldots)$ in a reference configuration; once all parameters are determined, the remaining coordinates of $(f, f_i, \ldots)$ are invariant.

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2. Classical invariant theory: The *invariant tensor theorem for* $O(n)$: invariants are

$$f, \quad f_i f_i, \quad f_{ii}, \quad f_{ij} f_i f_j, \quad f_{ij} f_{ij}, \quad f_{ijk} f_{ijk}, \ldots$$

How many invariants do we need? Is this invariant complete?

1. Fails to detect singular parts, e.g. $f(R) = \text{const.}$, $R \subset \mathbb{R}^2$, because all sub-parts of R are locally equivalent under $E(2)$.

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3. The signature $(f, f_i f_i, f_{ii}, f_{ij} f_i f_j) \subset \mathbb{R}^4$ locally determines f up to $E(2)$.

Another example: $SA(2)$

- Here the action is $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, $\det A = 1$.
- The prolonged action is $f \mapsto f$, $f_i \mapsto A_{ij} f_j$, $f_{ij} \mapsto A_{ik} A_{il} f_{kl}, \ldots$
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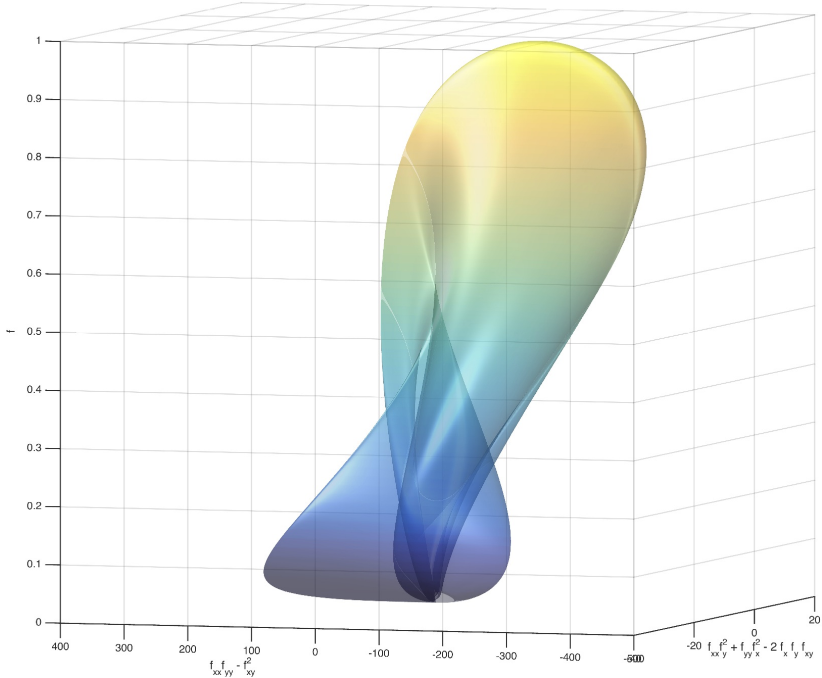
- Up to ternary: Alexander Bessel, 1869:
  - There are no invariants of 1st order
  - There are two invariants of 2nd order, the cubic ones are

$$\det f_{ij}, \quad f_{yy} f_x^2 + f_{xx} f_y^2 - 2 f_{xy} f_x f_y.$$

  - There are 15 independent polynomial invariants of 3rd order,

$$f_y f_{yy} f_{xxx} - 2 f_y f_{xy} f_{xxy} - f_x f_{yy} f_{xxy} + f_y f_{xx} f_{xyy} + 2 f_x f_{xy} f_{xyy} - f_x f_{xx} f_{yyy}$$

$$f_{yy} f_{xxy}^2 + f_{xx} f_{xyy}^2 + f_{xy} f_{xxx} f_{yyy} - f_{yy} f_{xxx} f_{xyy} - f_{xy} f_{xxy} f_{xyy} - f_{xx} f_{xxy} f_{yyy}$$



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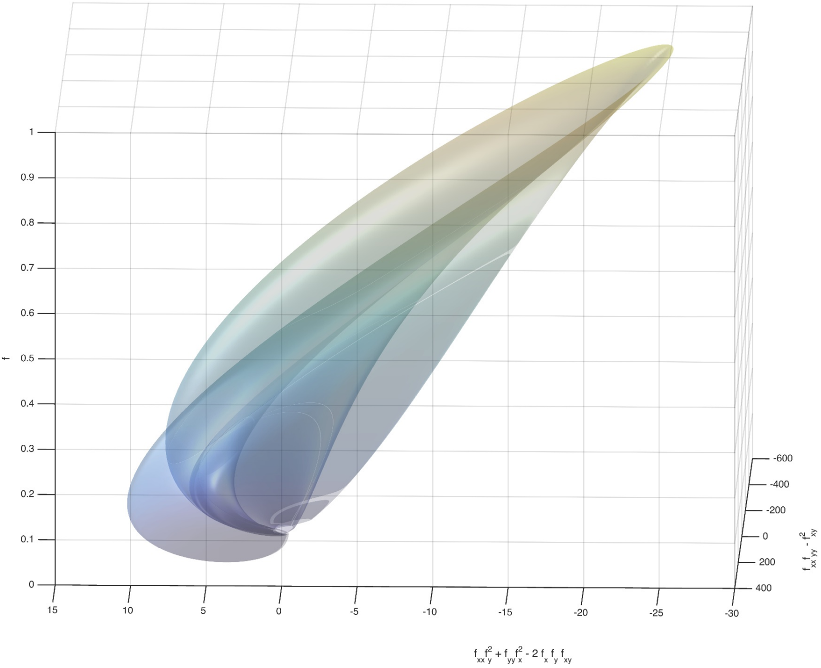

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-15
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-25
-30
$f_{xx} f_y^2 + f_{yy} f_x^2 - 2 f_x f_y f_{xy}$
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- But these are also invariant under the bigger group $\mathrm{Diff}_{\mathrm{vol}}(\mathbb{R}^2)$, so they can never be a complete invariant for $SA(2)$ – a hidden symmetry
- Thus the number of derivatives becomes an important quantity attached to each case.

derivatives needed for each group on k-colour images

		$k=1$	$k=2$	$k=3$
special Euclidean	$\mathrm{SE}(2)$	2	1	1
Euclidean	$\mathrm{E}(2)$	2	1	1
similarity	$\mathrm{Sim}(2)$	2	1	1
special affine	$\mathrm{SA}(2)$	2	2	2
affine	$\mathrm{A}(2)$	3	2	2
Möbius	$\mathrm{PSL}(2,\mathbb{C})$	3	3	3
projective	$\mathrm{PSL}(3,\mathbb{R})$	3	2	2
volume preserving	$\mathrm{Diff}_{\mathrm{vol}}$	–	1	1
conformal	$\mathrm{Diff}_{\mathrm{con}}$	3	1	1
all diffeos	Diff	–	–	0

Part II: Currents as a tool for shape space

Shape space is a space of the form

$$\mathrm{Imm}(M, N)/\mathrm{Diff}(M)$$

which is the image of the smooth immersions of M into N, forgetting the parameterization.

We will consider the images of oriented smooth planar curves,

$$\mathrm{Imm}(S^1, \mathbb{R}^2)/\mathrm{Diff}^+(\mathrm{S}^1).$$

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- Explicitly,

$$(1 - \nabla^2)\beta = \frac{\phi'(t)}{\|\phi'(t)\|}\delta_{\phi(S^1)}.$$

Currents: discrete side

Setup: $\phi\colon S^1 \to \mathbb{R}^2$, $[\phi](\alpha) = \int_{\phi(S^1)} \alpha$. Need:

1. A space V of finite elements on S^1;
2. A space W of finite elements on $\mathbb{R}^2$;
3. A quadrature approximation of $[\phi]|_W$; and
4. The dual norm restricted to W^*.

This setup yields a powerful, flexible, and robust way to work with shapes.

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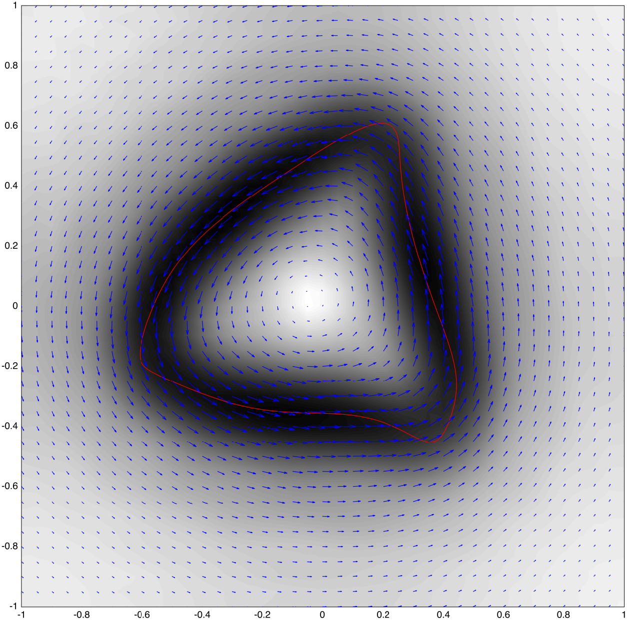
Specifically, we compute

$$G_{ij} = (w_i, w_j)_{H^1}$$

and solve

$$G_{ij}\beta_j = [\phi](w_i).$$

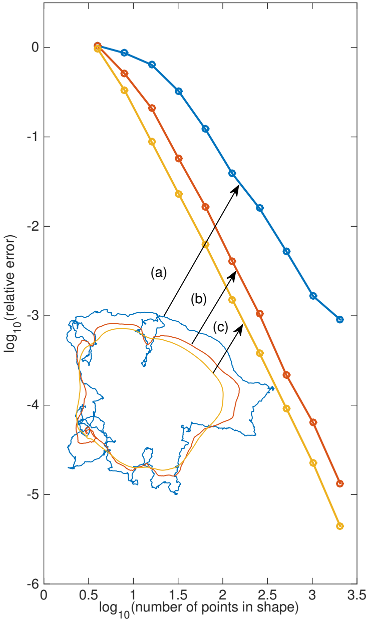
A shape and its representer



How accurate is it?

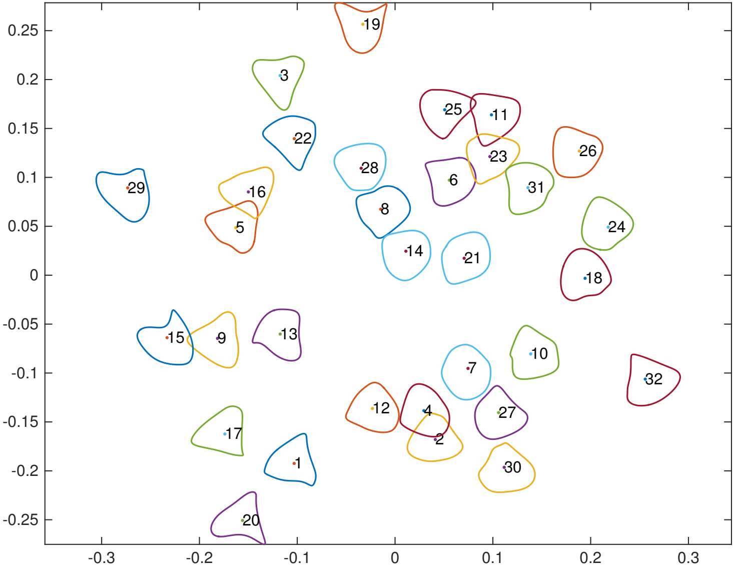
- The currents determine the shape very accurately: the error is $\mathcal{O}(h^5)$ for discontinuous quadratic elements.
- The induced metric is not very accurate, because the representers are only in H^1; errors are $\mathcal{O}(h)$.

Quadrature error on nonsmooth shapes



Shape
(a) is in C^0,
(b) is in $C^{1-\varepsilon}$,
(c) is in $C^{2-\varepsilon}$.

32 random shapes compared using finite element currents



Thank you for your attention!