Numerical discretisations of stochastic wave equations

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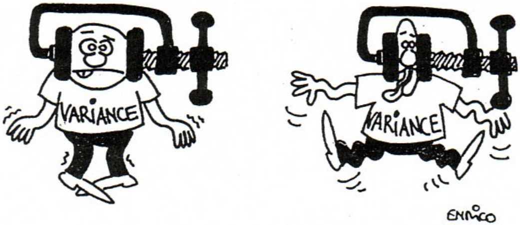
Outline

I. Crash course on SPDEs
II. The stochastic wave equation
III. Numerical discretisations
IV. Ongoing and future works

I. Crash course on SPDEs



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Motivation

We can see a stochastic wave equation ($(x,t) \in [0,1] \times [0,T]$)

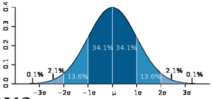
$$u_{tt}(x,t) - u_{xx}(x,t) = \text{RANDOM PERTURBATION}$$

as an infinite system of SDE (pseudo-spectral method)

$$\mathrm{d}\dot{X}_j(t) + \omega_j^2 X_j(t)\,\mathrm{d}t = \mathrm{d}\beta_j(t), \quad j \in \mathbb{Z},$$

where $\beta_j(t)$ are standard Brownian motions for $t \in [0,T]$:

- $\beta_j(0) = 0$ a.s.



- For $0 \le s < t \le T$ we have $\beta_j(t) - \beta_j(s) \sim N(0, t-s) = \sqrt{t-s}N(0,1)$.
- For $0 \le s \le t \le v \le w \le T$ the increments $\beta_j(t) - \beta_j(s)$ and $\beta_j(w) - \beta_j(v)$ are independent.

SPDEs: Notations and definitions

Mainly TWO approaches to define SPDEs:
Functional setting (SDE in Hilbert space) and random-field approach.
Let $\mathcal{D} \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a nice domain. Let us first consider the linear stochastic wave equation with additive noise in the Hilbert space $U := L_2(\mathcal{D})$:

$$\begin{aligned} \mathrm{d}\dot{u} - \Delta u \,\mathrm{d}t &= \mathrm{d}W && \text{in } \mathcal{D} \times (0, T), \\ u &= 0 && \text{in } \partial\mathcal{D} \times (0, T), \\ u(\cdot, 0) = u_0,\ \dot{u}(\cdot, 0) &= v_0 && \text{in } \mathcal{D}. \end{aligned}$$

Here $u = u(x, t)$ is a U-valued stochastic process, that is

$$u : [0, T] \times \Omega \to U = L_2(\mathcal{D}), u(t) := u(t, \omega) : \mathcal{D} \to \mathbb{R},$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is our probability space.
We will now define a Fourier series for the infinite dimensional Wiener process $W(t)$.

Definition of the noise

Let $Q \in \mathcal{L}(U)$ be a bounded, linear, symmetric, non-negative operator

$\Rightarrow$ The operator Q has eigenpairs $\{(\gamma_j, e_j)\}_{j=1}^{\infty}$ with orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of U.

Theorem. The Wiener process with covariance operator Q is given by

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} e_j \beta_j(t),$$

where $\beta_j(t)$ are i.i.d. standard Brownian motion.

The eig. values $\gamma_j > 0$ of the operator Q determine the spatial correlation of the noise.

Covariance operator

Recall ”definition” of noise: $W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} e_j \beta_j(t)$.

Consider two types of covariance operator:

- Cylindrical Wiener process, e.g. $Q = I$:

$$W(t) = \sum_{j=1}^{\infty} e_j \beta_j(t).$$

  Noise is white in space and time.

- Operator Q is trace-class if

$$\mathrm{Tr}(Q) = \sum_{j=1}^{\infty} \gamma_j < \infty.$$

  This gives noise with some spatial correlation.

Stochastic integrals

With this definition of the noise, we (*Da Prato, Zabczyk*, f.ex.) can define the stochastic Itô integral

$$\int_0^t \Phi(s)\,\mathrm{d}W(s)$$

together with Itô's isometry

$$\mathbb{E}\Big[\Big\|\int_0^t \Phi(s)\,\mathrm{d}W(s)\Big\|_U^2\Big] = \int_0^t \|\Phi(s)Q^{1/2}\|_{HS}^2\,\mathrm{d}s,$$

where we recall that $U = L_2(\mathcal{D})$ and the Hilbert-Schmidt norm on compact operators $\|T\|_{HS} := \mathrm{Tr}(TT^*) = \Big(\sum_{j=1}^{\infty}\|T\varphi_j\|_U^2\Big)^{1/2}$ with $\{\varphi_j\}_{j=1}^{\infty}$ an ON basis in U.

Last slide of the crash course . . .

Since we will deal with mean-square error bounds, the following norm will be useful

$$\|v\|_{L_2(\Omega,U)} := \mathbb{E}[\|v\|_U^2]^{1/2},$$

where we recall that $U = L_2(\mathcal{D})$ and that $\mathbb{E}$ is the mathematical expectation on our probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

No … this was not the last slide :-)

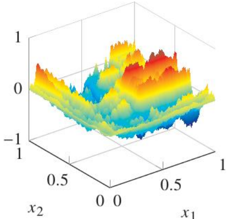
The second approach is based on another definition of the noise.
Problem (1d for simplicity):

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \dot{W}(t,x).$$

Here, $W(t,x)$ is a Brownian sheet (multi-parameter version of Brownian motion). That is, the noise term $\dot{W}(t,x)$ is a mean zero Gaussian noise with spatial correlation $k(\cdot,\cdot)$, i.e.

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)k(x,y),$$

where δ is a Dirac delta function at the origin.

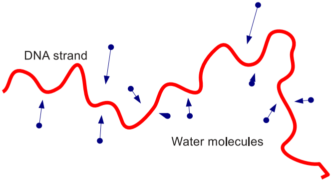


Thanks to G. Lord

II. The stochastic wave equation

Thanks to A. Grandchamp

Motivation: motion of DNA molecule in a liquid



Motion of a strand of DNA floating in a liquid (*Gonzalez, Maddocks* 2001, *Dalang* 2009):

DNA molecule $\rightsquigarrow$ string $\rightsquigarrow$ system of 3 wave equations in $\mathbb{R}^3$.

Liquid particles hit DNA $\rightsquigarrow$ stochastic motion.

The stochastic wave equation

Consider the stochastic wave equation

$$
\begin{aligned}
&\mathrm{d}\dot{u} - \Delta u \,\mathrm{d}t = f(u)\,\mathrm{d}t + g(u)\,\mathrm{d}W && \text{in } \mathcal{D} \times (0,T),\\
&u = 0 && \text{in } \partial\mathcal{D} \times (0,T),\\
&u(\cdot,0) = u_0,\ \dot{u}(\cdot,0) = v_0 && \text{in } \mathcal{D},
\end{aligned}
$$

where $u = u(x,t)$, $\mathcal{D} \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded convex domain with polygonal boundary $\partial\mathcal{D}$. The stochastic process $\{W(t)\}_{t\geq 0}$ is an $L_2(\mathcal{D})$-valued (possibly cylindrical) Q-Wiener process.

We set $\Lambda = -\Delta$ with $D(\Delta) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$.

Abstract formulation of the problem

Recall the problem: $\mathrm{d}\dot{u} - \Delta u \, \mathrm{d}t = f(u) \, \mathrm{d}t + g(u) \, \mathrm{d}W$ and $\Lambda = -\Delta$. Define the velocity of the solution $u_2 := \dot{u}_1 := \dot{u}$ and rewrite the above problem as

$$\begin{aligned} \mathrm{d}X(t) &= AX(t) \, \mathrm{d}t + F(X(t)) \, \mathrm{d}t + G(X(t)) \, \mathrm{d}W(t), \quad t > 0, \\ X(0) &= X_0, \end{aligned}$$

where $X := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $A := \begin{bmatrix} 0 & I \\ -\Lambda & 0 \end{bmatrix}$, $F(X) := \begin{bmatrix} 0 \\ f(u_1) \end{bmatrix}$, $G(X) := \begin{bmatrix} 0 \\ g(u_1) \end{bmatrix}$, and $X_0 := \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$.

The operator A is the generator of a strongly continuous semigroup $E(t) = \mathrm{e}^{tA}$.

Exact solution of the stochastic wave equation

The stochastic wave equation

$$\begin{aligned} \mathrm{d}X(t) &= AX(t)\,\mathrm{d}t + F(X(t))\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t), \quad t > 0, \\ X(0) &= X_0, \end{aligned}$$

has a unique mild solution (*recall* $E(t) = \mathrm{e}^{tA}$)

$$X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s))\,\mathrm{d}s + \int_0^t E(t-s)G(X(s))\,\mathrm{d}W(s).$$

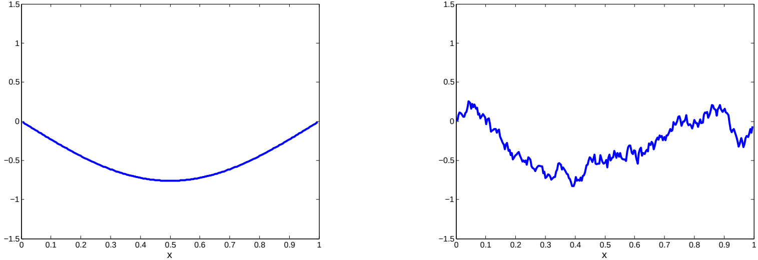
This is the variation-of-constants formula :-)

Obs. Here, one needs some regularity assumptions on the noise, f *and* g.

1d stochastic sine-Gordon

Problem: For $(x,t) \in [0,1] \times [0,1]$, consider the SPDE with space-time white-noise

$$
\begin{aligned}
&\mathrm{d}\dot{u} - \Delta u \,\mathrm{d}t = -\sin(u)\,\mathrm{d}t - \sin(u)\,\mathrm{d}W, \\
&u(0,t) = u(1,t) = 0, \\
&u(x,0) = \cos(\pi(x - 1/2)), \;\; \dot{u}(x,0) = 0.
\end{aligned}
$$



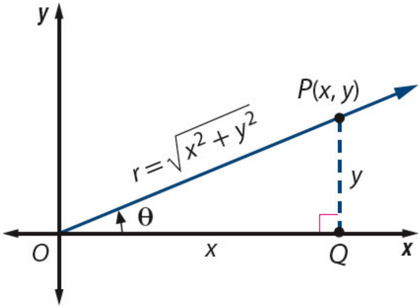
sine-Gordon

III. Numerical discretisations

tangent of $\theta = \tan\theta = \dfrac{y}{x}\ (x \neq 0)$

sine of $\theta = \sin\theta = \dfrac{y}{r}$

cosine of $\theta = \cos\theta = \dfrac{x}{r}$



Thanks to www.images.google.com

Results (I) ... spoiler alert ...

Problem: $\mathrm{d}\dot{u} - \Delta u \,\mathrm{d}t = \mathrm{d}W$.

Spatial discretisation by standard linear FEM with mesh h.

Time discretisation by stochastic trigonometric method with step size k.

Theorems (*C.*, *Larsson*, *Sigg* 2013)

- Exact linear growth of the energy for the num. solution.
- Mean-square order of convergence at most one (depends on the regularity of the noise)

$$\|U_1^n - u_{h,1}(t_n)\|_{L_2(\Omega,U)} \leq Ck^{\min\{\beta,1\}} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}},$$

  where $u_{h,1}$ is the FE solution of our problem.
- Mean-square errors for the full discretisation.

FEM done in Kovács, Larsson, Saedpanah 2010

Results (II)

Problem: $\mathrm{d}\dot{u} - \Delta u \, \mathrm{d}t = f(u) \, \mathrm{d}t + g(u) \, \mathrm{d}W$.

Spatial discretisation by standard linear FEM with mesh h.

Time discretisation by stochastic trigonometric method with step size k.

Theorems (*Anton*, *C*., *Larsson*, *Wang* 2016)

- Mean-square errors for the full discretisation

$$
\begin{aligned}
\|U_1^n - u_1(t_n)\|_{L_2(\Omega,U)} &\leq C \cdot \left(h^{\frac{2\beta}{3}} + k^{\min(\beta,1)}\right) \text{ for } \beta \in [0,3], \\
\|U_2^n - u_2(t_n)\|_{L_2(\Omega,U)} &\leq C \cdot \left(h^{\frac{2(\beta-1)}{3}} + k^{\min(\beta-1,1)}\right) \text{ for } \beta \in [1,4].
\end{aligned}
$$

- Almost linear growth of the energy for the num. solution of the nonlinear problem with additive noise.

Results (III) for random-field

Problem $\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(u(t,x)) + \sigma(u(t,x))\frac{\partial^2 W}{\partial x \partial t}(t,x)$.

Spatial discretisation by centered FD with mesh Δx.

Time discretisation by stochastic trigonometric method with step size Δt.

Theorem (*C.*, *Quer-Sardanyons* 2015). Consider, for simplicity, the initial values $u_0 = v_0 = 0$. Assume f and σ satisfy a global Lipschitz and linear growth condition.
Let $p \geq 1$. Then, the following estimate of the error for the full discretisation holds:

$$\sup_{(t,x)\in[0,T]\times[0,1]} \left(\mathbb{E}\left[\left|u^{M,N}(t,x) - u(t,x)\right|^{2p}\right]\right)^{\frac{1}{2p}} \leq C_1 \left(\Delta x\right)^{\frac{1}{3}-\varepsilon} + C_2 \left(\Delta t\right)^{\frac{1}{2}},$$

for all small enough $\varepsilon > 0$. The constants C_1 and C_2 are positive and do not depend neither on M nor on N.

Stochastic trigonometric method (Hilb. sp.) (I)

Recall: The FE problem for the linear wave equation with additive noise reads ($\mathcal{P}_h$ orthogonal projection, $B = [0, I]$)

$$\mathrm{d}X_h(t) = A_h X_h(t)\,\mathrm{d}t + \mathcal{P}_h B\,\mathrm{d}W(t), \quad X_h(0) = X_{h,0}, \quad t > 0.$$

Use variation-of-constants formula for the exact solution:

$$X_h(t) = E_h(t)X_{h,0} + \int_0^t E_h(t-s)\mathcal{P}_h B\,\mathrm{d}W(s),$$

where

$$E_h(t) = \mathrm{e}^{tA_h} = \begin{bmatrix} C_h(t) & \Lambda_h^{-1/2} S_h(t) \\ -\Lambda_h^{1/2} S_h(t) & C_h(t) \end{bmatrix}$$

with $C_h(t) = \cos(t\Lambda_h^{1/2})$ and $S_h(t) = \sin(t\Lambda_h^{1/2})$.

Stochastic trigonometric method (II)

Discretise the stochastic integral in the var.-of-const. formula

$$X_h(k) = E_h(k)X_{h,0} + \int_0^k E_h(k-s)\mathcal{P}_h B \,\mathrm{d}W(s)$$

to obtain $U^{n+1} = E_h(k)U^n + E_h(k)\mathcal{P}_h B\Delta W^n$, that is,

$$\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \end{bmatrix} = \begin{bmatrix} C_h(k) & \Lambda_h^{-1/2}S_h(k) \\ -\Lambda_h^{1/2}S_h(k) & C_h(k) \end{bmatrix} \begin{bmatrix} U_1^n \\ U_2^n \end{bmatrix} + \begin{bmatrix} \Lambda_h^{-1/2}S_h(k) \\ C_h(k) \end{bmatrix} \mathcal{P}_h\Delta W^n,$$

where $\Delta W^n = W(t_{n+1}) - W(t_n)$ denotes the Wiener increments. Get an approximation $U_j^n \approx u_{h,j}(t_n)$ of the exact solution of our FE problem at the discrete times $t_n = nk$.

Similar ideas used for the semi-linear case with multiplicative noise.

Chap. XIII of the yellow bible by Hairer, Lubich, Wanner 2006

A trace formula

Theorem (*C., Larsson, Sigg* 2013) Expected value of the energy of the FE solution $X_h(t) = [u_{h,1}(t), u_{h,2}(t)]$ grows linearly with time:

$$
\begin{aligned}
& \mathbb{E}\Big[\tfrac{1}{2}\big(\|\Lambda_h^{1/2} u_{h,1}(t)\|^2_{L_2(\mathcal{D})} + \|u_{h,2}(t)\|^2_{L_2(\mathcal{D})}\big)\Big] \\
= \; & \mathbb{E}\Big[\tfrac{1}{2}\big(\|\Lambda_h^{1/2} u_{h,0}\|^2_{L_2(\mathcal{D})} + \|v_{h,0}\|^2_{L_2(\mathcal{D})}\big)\Big] + \tfrac{1}{2} t \,\mathrm{Tr}(\mathcal{P}_h Q \mathcal{P}_h).
\end{aligned}
$$

We have the same result for the num. sol. given by the stochastic trigonometric method:

$$
\begin{aligned}
& \mathbb{E}\Big[\tfrac{1}{2}\big(\|\Lambda_h^{1/2} U_1^n\|^2_{L_2(\mathcal{D})} + \|U_2^n\|^2_{L_2(\mathcal{D})}\big)\Big] \\
= \; & \mathbb{E}\Big[\tfrac{1}{2}\big(\|\Lambda_h^{1/2} u_{h,0}\|^2_{L_2(\mathcal{D})} + \|v_{h,0}\|^2_{L_2(\mathcal{D})}\big)\Big] + \tfrac{1}{2} t_n \,\mathrm{Tr}(\mathcal{P}_h Q \mathcal{P}_h).
\end{aligned}
$$

Obs. These are long-time results for the numerical solutions.

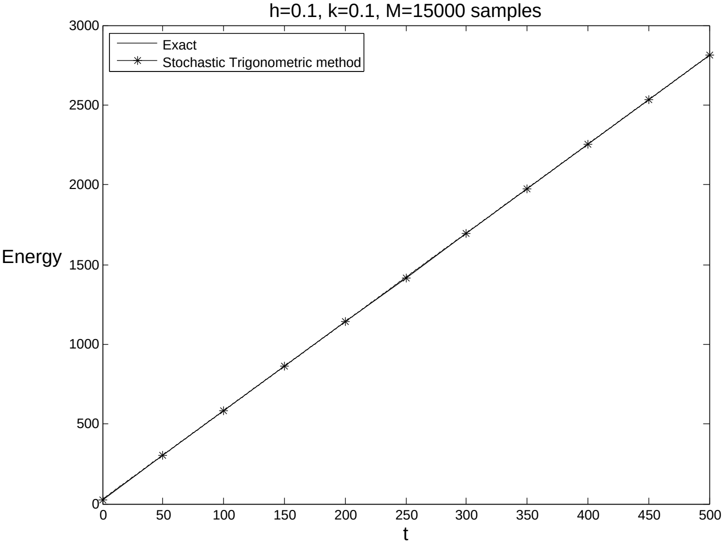
A trace formula: Numerical experiments (I)

We consider the linear stochastic wave equation

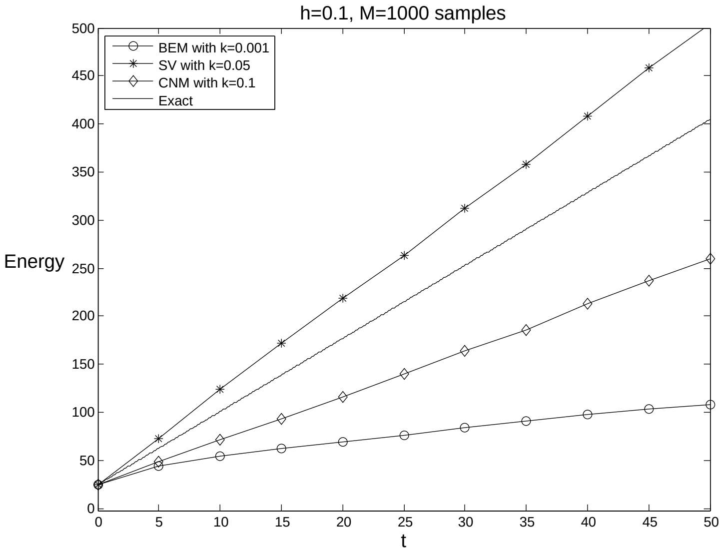
$$\begin{aligned}
&\mathrm{d}\dot{u} - \Delta u \,\mathrm{d}t = \mathrm{d}W, \quad (x,t) \in (0,1) \times (0,1), \\
&u(0,t) = u(1,t) = 0, \quad t \in (0,1), \\
&u(x,0) = \cos(\pi(x-1/2)),\ \dot{u}(x,0) = 0, \quad x \in (0,1),
\end{aligned}$$

with a Q-Wiener process $W(t)$ with covariance operator $Q = \Lambda^{-1/2}$. Where we recall $\Lambda = -\Delta$.

A trace formula: Numerical experiments (II)



A trace formula: Numerical experiments (III)



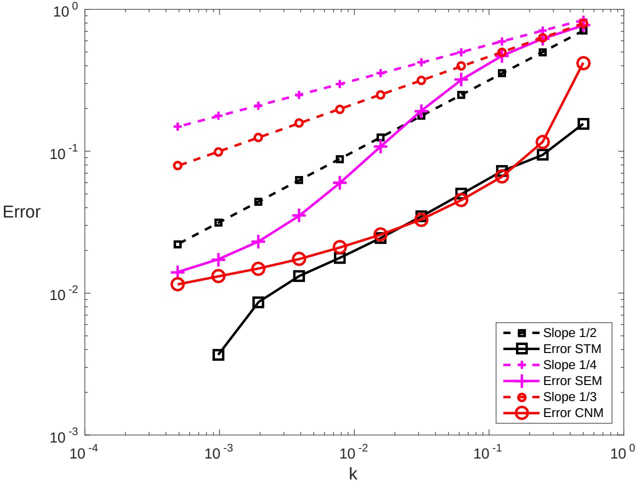
Mean-square errors: Numerical experiments (I)

We consider the 1d sine-Gordon equation driven by a multiplicative space-time white noise ($(x,t) \in (0,1) \times (0,0.5)$)

$$\begin{aligned}
&\mathrm{d}\dot{u}(x,t) - \Delta u(x,t)\,\mathrm{d}t = -\sin(u(x,t))\,\mathrm{d}t + u(x,t)\,\mathrm{d}W(x,t),\\
&u(0,t) = u(1,t) = 0, \quad t \in (0,0.5),\\
&u(x,0) = \cos(\pi(x-1/2)),\ \dot{u}(x,0) = 0, \quad x \in (0,1),
\end{aligned}$$

with a space-time white noise with $Q = I$ ($\beta < 1/2$).

Mean-square errors: Numerical experiments (II)



Parameters: "True sol." with STM with $k_{\text{exact}} = 2^{-11}$ *and* $h_{\text{exact}} = 2^{-9}$. $M_{\text{s}} = 2500$ *for the expectations.*

IV. Ongoing and future works

@ Universal Pictures

Ongoing and future works (I)

With R. Anton (Umeå University).

Exponential methods for the time discretisation of stochastic Schrödinger equations

$$\begin{aligned} \mathrm{i}\mathrm{d}u - (\Delta u + F(u))\mathrm{d}t &= G(u)\mathrm{d}W \quad \text{in} \quad \mathbb{R}^d \times [0,T] \\ u(0) &= u_0, \end{aligned}$$

where $u = u(x,t)$, with $t \geq 0$ and $x \in \mathbb{R}^d$, is a complex valued random process.

With R. Anton and L. Quer-Sardanyons (UA Barcelona).

Exponential methods for parabolic problems (random-field approach).

Ongoing and future works (II)

With G. Dujardin (Inria Lille Nord-Europe).

Exponential integrators for nonlinear Schrödinger equations with white noise dispersion

$$\begin{gathered} \mathrm{i}\mathrm{d}u + c\Delta u \circ \mathrm{d}\beta + |u|^{2\sigma} u \, \mathrm{d}t = 0 \\ u(0) = u_0, \end{gathered}$$

where $u = u(x,t)$, with $t \geq 0$ and $x \in \mathbb{R}^d$, is a complex valued random process; c and σ are positive real numbers; and $\beta = \beta(t)$ is a real valued Brownian motion.

Look for mean-square error estimates and mass-preserving exponential integrators.

Thanks to www.images.google.com

Main ingredients of the proofs

Recall:

$$
\begin{aligned}
u^{M,N}(t,x) = & \int_0^t\!\!\int_0^1 \left\{ G^M(t-\kappa_N^T(s),x,y) f\left(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\right)\right\} \mathrm{d}y\,\mathrm{d}s \\
& + \int_0^t\!\!\int_0^1 \left\{ G^M(t-\kappa_N^T(s),x,y) \sigma\left(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\right)\right\} W(\mathrm{d}s,\mathrm{d}y).
\end{aligned}
$$

1. Write $u^{M,N}(t,x) - u(t,x) = u^{M,N}(t,x) - u^M(t,x) + u^M(t,x) - u(t,x)$ and use the spatial error estimate from *Quer-Sardanyons, Sanz-Solé* 2006 to get the order $C_1\,(\Delta x)^{\frac{1}{3}-\varepsilon}$ for all small enough $\varepsilon > 0$.
2. Next, consider $u^{M,N}(t,x) - u^M(t,x)$ and use a Gronwall argument.
3. Use properties of $G^M(t,x,y)$, of $u^M(t,x)$, of $u^{M,N}(t,x)$, assumptions on f and σ, Hölder's inequality, Burkholder-Davis-Gundy's inequality and finally Gronwall's inequality to bound the error of the fully-discrete solution.

Main steps for the proofs (I)

First, by definition of u^M and $u^{M,N}$, consider the difference

$$
\begin{aligned}
u^{M,N}(t,x) - u^M(t,x) = & \int_0^t\!\!\int_0^1 \Big\{ G^M(t-\kappa_N^T(s),x,y) f\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big) \\
& - G^M(t-s,x,y) f\big(u^M(s,\kappa_M(y))\big) \Big\} \, \mathrm{d}y \, \mathrm{d}s \\
& + \int_0^t\!\!\int_0^1 \Big\{ G^M(t-\kappa_N^T(s),x,y) \sigma\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big) \\
& - G^M(t-s,x,y) \sigma\big(u^M(s,\kappa_M(y))\big) \Big\} \, W(\mathrm{d}s,\mathrm{d}y).
\end{aligned}
$$

Next, add and subtract some terms in order to be able to use the properties of f and σ.

Main steps for the proofs (II)

The first expression

$$u^{M,N}(t,x) - u^M(t,x) = \int_0^t\!\!\int_0^1 \Big\{ G^M(t-\kappa_N^T(s),x,y) f\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big) - G^M(t-s,x,y) f\big(u^M(s,\kappa_M(y))\big) \Big\}\,\mathrm{d}y\,\mathrm{d}s + \text{noisy blabla}$$

can be decomposed as the sum of 3 terms:

$$D_1 := \int_0^t\!\!\int_0^1 G^M(t-\kappa_N^T(s),x,y) \times \Big\{ f\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big) - f\big(u^M(\kappa_N^T(s),\kappa_M(y))\big) \Big\}\,\mathrm{d}y\,\mathrm{d}s,$$

$$D_2 := \int_0^t\!\!\int_0^1 \Big\{ G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y) \Big\} \times f\big(u^M(\kappa_N^T(s),\kappa_M(y))\big)\,\mathrm{d}y\,\mathrm{d}s,$$

$$D_3 := \int_0^t\!\!\int_0^1 G^M(t-s,x,y) \Big\{ f\big(u^M(\kappa_N^T(s),\kappa_M(y))\big) - f\big(u^M(s,\kappa_M(y))\big) \Big\}\,\mathrm{d}y\,\mathrm{d}s.$$

Main steps for the proofs (III)

The second expression

$$\int_0^t\int_0^1 \left\{G^M(t-\kappa_N^T(s),x,y)\,\sigma\left(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\right) - G^M(t-s,x,y)\,\sigma\left(u^M(s,\kappa_M(y))\right)\right\} W(\mathrm{d}s,\mathrm{d}y).$$

can be decomposed as the sum of 3 terms:

$$D_4 := \int_0^t\int_0^1 G^M(t-\kappa_N^T(s),x,y) \times \left\{\sigma\left(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\right) - \sigma\left(u^M(\kappa_N^T(s),\kappa_M(y))\right)\right\} W(\mathrm{d}s,\mathrm{d}y),$$

$$D_5 := \int_0^t\int_0^1 \left\{G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y)\right\} \times \sigma\left(u^M(\kappa_N^T(s),\kappa_M(y))\right) W(\mathrm{d}s,\mathrm{d}y),$$

$$D_6 := \int_0^t\int_0^1 G^M(t-s,x,y) \times \left\{\sigma\left(u^M(\kappa_N^T(s),\kappa_M(y))\right) - \sigma\left(u^M(s,\kappa_M(y))\right)\right\} W(\mathrm{d}s,\mathrm{d}y).$$

Main steps for the proofs (IV)

We now have to estimate each of the terms $D_1, \ldots, D_6$.
Estimates for D_6:

$$D_6 := \int_0^t\int_0^1 G^M(t-s,x,y) \times \left\{\sigma\left(u^M(\kappa_N^T(s),\kappa_M(y))\right) - \sigma\left(u^M(s,\kappa_M(y))\right)\right\} W(\mathrm{d}s,\mathrm{d}y).$$

By Burkholder-Davis-Gundy's and Hölder's inequalities, and Lipschitz condition on σ, one obtains

$$\mathbb{E}\left[\left|D_6\right|^{2p}\right] \leq C\int_0^t\int_0^1 G^M(t-s,x,y)^2\,\mathrm{d}y \sup_{x\in[0,1]} \mathbb{E}[|u^M(\kappa_N^T(s),x) - u^M(s,x)|^{2p}]\,\mathrm{d}s.$$

The Hölder continuity property of the process $u^M(t,x)$ gives

$$\mathbb{E}\left[\left|D_6\right|^{2p}\right] \leq C\int_0^t (\kappa_N^T(s)-s)^p\,\mathrm{d}s \leq C\,(\Delta t)^p.$$

Main steps for the proofs (V)

All these estimates together give

$$\sup_{(t,x)\in[0,T]\times[0,1]} \mathbb{E}\left[\left|u^{M,N}(t,x) - u^{M}(t,x)\right|^{2p}\right] \leq C_1\,(\Delta t)^p$$

$$+\, C_2 \int_0^t \sup_{(r,x)\in[0,s]\times[0,1]} \mathbb{E}\left[\left|u^{M,N}(r,x) - u^{M}(r,x)\right|^{2p}\right] \mathrm{d}s,$$

for some positive constants C_1 and C_2 independent of N and M.
An application of Gronwall's lemma finally lead to

$$\sup_{(t,x)\in[0,T]\times[0,1]} \left(\mathbb{E}\left[\left|u^{M,N}(t,x) - u^{M}(t,x)\right|^{2p}\right]\right)^{\frac{1}{2p}} \leq C\,(\Delta t)^{\frac{1}{2}}.$$

This concludes the proof of the theorem.

Thanks for your attention!!