Nonrelativistic limit of the Nonlinear Klein-Gordon equation: dynamics over long times

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Outline

Summary

- The problem: Nonrelativistic limit of the Klein Gordon equation

$$\frac{1}{c^2}u_{tt} - \Delta u + c^2 u = -\lambda u^3 \ , \quad x \in M \ , \quad c \to \infty$$

  M compact manifold or $\mathbb{R}^d$.
- Heuristic Discussion
  - Formal limit
  - Formal limit, how to estimate the error

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  - Hamiltonian formulation
  - Normal Form: standard approach/Galerkin Averaging
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- Longer time estimates
  - On $\mathbb{R}^3$ by dispersive estimates
  - On $[0, \pi]$ a preliminary result by extension of BNF for semilinear PDEs

Outline

- The operator and the complex variables $v := \dot{u}/c^2$

$$\langle \nabla \rangle_c := \left(c^2 - \Delta\right)^{1/2} = c - \frac{1}{2c}\Delta + \mathcal{O}\left(\frac{\Delta^2}{c^3}\right)$$

Formal manipulations

- The operator and the complex variables $v := \dot{u}/c^2$

$$\langle\nabla\rangle_c := \left(c^2 - \Delta\right)^{1/2} = c - \frac{1}{2c}\Delta + \mathcal{O}\left(\frac{\Delta^2}{c^3}\right)$$

$$\psi = \frac{1}{\sqrt{2}}\left[\left(\frac{\langle\nabla\rangle_c}{c}\right)^{1/2} u - i\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2} v\right] = \frac{u - iv}{\sqrt{2}} + h.o.t.$$

- Structure of the equations:

$$\dot{\psi} = ic\,\langle\nabla\rangle_c \psi + N(\psi) \tag{1}$$

with

$$N(\psi) := +\lambda\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}\left(\frac{\psi + \bar{\psi}}{\sqrt{2}}\right)^3$$

nonlinear term.

- A model problem

$$\dot{\psi} = ic\,\langle\nabla\rangle_c\psi + \tilde{N}(\psi)\ , \quad \tilde{N}(\psi) := \lambda\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}|\psi|^3$$

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- Approximate:

$$-i\dot{\psi}_a = c^2\psi_a - \frac{1}{2}\Delta\psi_a + \lambda|\psi_a|^3 \tag{2}$$

- Gauge transform: $\psi_a(t) := e^{ic^2t}\phi(t)$ then

$$-i\dot{\phi} = -\frac{1}{2}\Delta\phi + \lambda|\phi|^3 \tag{3}$$

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- A solution of (2) solves actually

$$\dot{\psi}_a = ic\,\langle\nabla\rangle_c\psi_a + \tilde{N}(\psi_a) + \frac{1}{c^2}R(t)\ ,$$
$$\frac{R(t)}{c^2} = (ic^2\psi_a - i\frac{1}{2}\Delta\psi_a + ic\,\langle\nabla\rangle_c\psi_a) + \left(\lambda\,|\psi_a|^3 - \tilde{N}(\psi_a)\right)\ .$$

- Equation for the error

$$\delta := \psi - \psi_a \ , \quad \dot{\delta} = ic \left\langle \nabla \right\rangle_c \delta + \tilde{N}(\psi_a + \delta) - \tilde{N}(\psi_a) - \frac{R(t)}{c^2}$$

- Duhamel

$$\delta(t) = \int_0^t e^{ic(t-s)\langle\nabla\rangle_c} [d\tilde{N}(\psi_a(s))] \ \delta(s) ds + \mathcal{O}(\delta^2) + \mathcal{O}(\frac{1}{c^2})$$

- solution $\|\delta(t)\| \leq c^{-2}$ for $|t| \lesssim 1$

Estimates of the error

- Equation for the error

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$$\dot{\delta} = ic \left\langle \nabla \right\rangle_c \delta + [d\tilde{N}(\psi_a(s))] \ \delta \quad \Longrightarrow \quad \|\delta(t)\| \leq e^{at} \delta_0$$

- In $\mathbb{R}^3$

$$\dot{\psi} = i\Delta\psi \Longrightarrow \psi(t) = \frac{c}{t^{3/2}} \int_{R^3} e^{\frac{|x-y|^2}{it}} \psi_0(y)dy$$

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- Strichartz estimates:

$$\left\| e^{it\Delta}\psi_0 \right\|_{L^2_t L^6_x} \lesssim \|\psi_0\|_{L^2_x}$$

- Stable under perturbation: They hold for

$$-i\dot{\psi} = -\Delta\psi + \epsilon A(t)\psi$$

for A in suitable classes.

Outline

- With loss of regularity, Tsutsumi 1984, Najman 1990,
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Let $\psi_c^0 \to \phi_0$ in $H^{1/2}$. Consider the solution $\phi(t)$ of NLS with $\phi(0) = \phi^0$, and let T^* be its maximal existence time. Let $\psi_c(t)$ be the solution of NLKG and let T_c^* be its maximal existence time, then

$$\liminf_{c\to\infty} T_c^* \geq T^*$$

and

$$\psi_c - e^{ic^2 t}\phi \to 0 \text{ , in } C([0, T^*), H^{1/2})$$

Use of adapted Strichartz estimates in c dependent Besov spaces.

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- Realistic models Mauser and collaborators around 2002

Theorem (Faou Schratz 2014)

Fix $T < T^*$, then for any s there exists s_1, s.t. if

$$\left\|\psi_c^0 - \phi^0\right\|_{H^{s+s_1}} \preceq c^{-1},$$

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Actually the result is stronger: Error$=\mathcal{O}(c^{-r})$, but not longer times!

Outline

The Hamiltonian

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$$H = \int_M c \left| \langle\nabla\rangle_c^{1/2} \psi \right|^2 + \frac{\lambda}{4} \left[\left(\frac{c}{\langle\nabla\rangle_c} \right)^{1/2} \frac{\psi + \bar{\psi}}{\sqrt{2}} \right]^4 dx$$

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- Expansion

$$H = \int_M c^2 |\psi|^2 dx + \int_M \left(\frac{1}{2} |\nabla \psi|^2 + \frac{\lambda}{2} \left(\psi + \bar{\psi} \right)^4 \right) dx$$
$$+ \text{singular } h.o.t.$$

- Rescale time: $\tau := c^2 t$

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- Structure: $\epsilon := c^{-2}$

$$H \sim h_0 + \sum_{k\geq 1} \epsilon^k h_k + \sum_{k\geq 1} \epsilon^k F_k \ .$$

with h_0 generating a periodic flow Φ^t.

Formal BNF

- Formal theory: $\forall r \geq 0$, $\exists \mathcal{T}^{(r)}$ formal canonical transformation s.t.

$$H \circ \mathcal{T}^{(r)} = h_0 + \epsilon(h_1 + \langle F_1 \rangle) + \sum_{k=2}^{r} \epsilon^r Z_r + \mathcal{O}(\epsilon^{r+1}) \ ,$$

with

$$\{h_0; Z_r\} = 0 \ , \quad \langle F_1 \rangle (\psi) := \frac{1}{2\pi} \int_0^{2\pi} F_1(\Phi^t \psi) dt \ .$$

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- the method is constructive, e.g. (for NLKG): $h_0 + \epsilon(h_1 + \langle F_1 \rangle) \equiv$NLS,

$$\epsilon^2(h_2 + Z_2) = \frac{1}{c^4} \int \left[\frac{1}{8} \psi \cdot \Delta^2 \bar{\psi} - 17\lambda^2 |\psi|^6 + \frac{3}{2} \lambda |\psi|^2 (\bar{\psi}\, \Delta \psi + \psi, \Delta \bar{\psi}) \right] dx$$

- Topology: $H^s(M)$, s large. Assume $X_{h_j} \in C^\infty(H^{s+2j}, H^s)$, $X_{F_j} \in C^\infty(H^{s+2(j-1)}, H^s)$, (true for NLKG)
- Cutoff operator Π_N :=spectral projector of $-\Delta$, on eigenvalues smaller then N^2
- Strategy:
  - cutoff the Hamiltonian: $H_N := H \circ \Pi_N$: the error is small as an operator loosing many derivatives
  - Put in normal form H_N, choose N and the loss of smoothness in a suitable way.

Let $B_s(R)$= Ball of radius R and center 0 in $H^s(M)$.

Theorem

Consider the Klein Gordon equation; fix $r \geq 1$ and $s \gg 1$. If $\epsilon \equiv c^{-2} \ll 1$, then there exists $\mathcal{T}^{(r)} : B_{4r^2+s}(1) \to B_{4r^2+s}(2)$ analytic, s.t.

$$H \circ \mathcal{T}^{(r)} = h_0 + \epsilon(h_1 + \langle F_1 \rangle) + \sum_{k=2}^{r} \epsilon^r Z_r + \epsilon^{r+1/2}\mathcal{R} \ ,$$

Furthermore, on $B_{s+4r^2}(1)$ one has

$$\|X_{Z_r}\|_s \, , \|X_{\mathcal{R}}\|_s \leq C$$

What about the dynamics?

Outline

Dynamics: general (reduction to Gronwall lemma)

- Simplified system: $H_{simp} := h_0 + \epsilon\big(h_1 + \langle F_1\rangle\big) + \sum_{k=2}^{r} \epsilon^r Z_r$

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$$\dot{\psi}_s = X_{H_{simp}}(\psi_s) = \text{NLS+h.o. normalised corrections}$$

then $\psi_a(t) := \mathcal{T}^{(r)}(\psi_s(c^2 t))$ solves

$$\dot{\psi}_a = ic \left\langle \nabla \right\rangle_c \psi_a + N(\psi_a) - \frac{1}{c^{2r}} \mathcal{T}^{(r)*} \mathcal{R}(\psi_a) = NLKG + \mathcal{O}(\epsilon^{-2r})$$

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- the remainder is evaluated on the **approximate solution** .
- If ψ is a solution of NLKG, then the error $\delta := \psi - \psi_a$ fulfills

$$\dot{\delta} = ic \langle \nabla \rangle_c \delta + [N(\psi_a + \delta) - N(\psi_a)] + \frac{1}{c^{2r}} \mathcal{T}^{(r)*} \mathcal{R}(\psi_a(t))$$

or (rescaling time to $t' = ct$)

$$\delta(t) = \frac{1}{c} \int_0^t e^{i(t-s)\langle \nabla \rangle_c} dN(\psi_a(s)) \delta(s) ds + \mathcal{O}(\delta^2) + \mathcal{O}(\frac{1}{c^{2r+1}})$$

Corollary

Fix s, assume $\|\psi_0\|_{4r^2+s} \le 1/2$ and

$$\exists T \ s.t. \ \|\psi_s(t)\|_{4r^2+s} \le 1 \ , \quad |t| \le T \tag{4}$$

(non rescaled time) then

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- Problem: the correction of second order become effective after a time $\mathcal{O}(c)$, so that they are here invisible.
- In the focusing case, the second order correction are defocusing, so they are expected to change qualitatively the dynamics of the normalized equation. What about the original system?

Outline

- Case $M = \mathbb{R}^3$: use Strichartz estimates to estimate sol of

$$\delta(t) = \frac{1}{c}\int_0^t e^{i(t-s)\langle\nabla\rangle_c} dN(\psi_a(s))\delta(s)ds \ \ :$$

Strichartz estimates typically persist under perturbations! (In suitable classes.)

- Standard estimates: let (p, q) be a Schrödinger admissible pair, then

$$\left\| e^{i\langle\nabla\rangle t}\psi \right\|_{L^p_t L^q_x} \lesssim \left\| \langle\nabla\rangle^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}\psi \right\|_{L^2_x} \ , \quad \langle\nabla\rangle := \sqrt{1-\Delta}$$

$$\left\| e^{i\Delta t}\psi \right\|_{L^p_t L^q_x} \lesssim \|\psi\|_{L^2_x}$$

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- Difficulty: in NLKG there is a change of smoothness, not in the limit equation NLS. No trivial uniform estimate is possible!
- Solution: scaling from the standard estimate by D'Ancona-Fanelli.

Lemma

For any Schrödinger admissible pair (p, q) and any $k \geq 0$, one has

$$\left\| e^{i\langle\nabla\rangle_c t}\psi \right\|_{L^p_t W^{k,q}_x} \lesssim c^{\frac{1}{q}-\frac{1}{2}} \left\| \langle\nabla\rangle_c{}^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}\psi \right\|_{H^k} \ . \tag{5}$$

Lemma

Take an initial datum such that the solution ψ_s of the normalized equation exists for all times and has the structure

$$\psi_s(x,t) = \psi_{rad}(x,t) + \sum_{l=1}^{N} \eta_l(x - v_l t) \qquad (6)$$

with some $\eta_l \in \mathcal{S}$, $v_l \in \mathbb{R}^3$ and $\psi_{rad} \in L^p_t W^{k,q}_x$ with (p,q) any Schrödinger admissible pair, then the flow map of

$$i \langle \nabla \rangle_c + \frac{1}{c} dN(\psi_s(t))$$

fulfills the estimate (5).

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fulfills the estimate (5).

Use this lemma to estimate

$$\left\| \frac{\langle \nabla \rangle_c^{1/2}}{c^{1/2}} \delta \right\|_{L^\infty_t H^k_x}$$

Theorem

Fix $k \geq 0$ and a large r , then there exists a large k_*, with the following property: take an initial datum such that the solution ψ_s of the normalized equation exists for all times and has the structure (6) and in particular $\psi_{rad} \in L^p_t W^{k_*,q}_x$; denote $\psi_a(t) := \mathcal{T}^{(r)}(e^{ic^2t}\psi_s(t))$. Let $\psi(t)$ be the sol of NLKG with the corresponding initial datum, then one has

$$\|\psi_a(t) - \psi(t)\|_{H^k} \lesssim \frac{1}{c} \ , \quad |t| \lesssim c^r \ , \quad c \gg 1$$

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$$\|\psi_a(t) - \psi(t)\|_{H^k} \lesssim \frac{1}{c} \ , \quad |t| \lesssim c^r \ , \quad c \gg 1$$

- Difficulty: do there exist solutions with the above property? Soliton resolution conjecture! Something is known, but not too much.
- The Theorem says that in this case the soliton dynamics and the dynamics of the radiation is well described by the approximate equation.

Outline

- A modified problem

$$\frac{1}{c^2}u_{tt} - u_{xx} + V * u + c^2 u = \lambda u \ .$$

- $V(x) = \sum_{k>0} \frac{V_k}{k^2}\cos(kx)$, $V_k \in [-1/2, 1/2]$ iid . $\mathcal{V}$ corresponding probability space endowed by the product measure.

Theorem

There exists $\mathcal{A} \subset (\mathcal{V} \times \mathbb{R})$ with

$$|\mathcal{A} \cap ([N, N+1] \times \mathcal{V})| = 1$$

s.t. the following holds true. Fix $\alpha > 0$ and $r \gg 1$ and take $(c, V) \in \mathcal{A}$, then there exists s_* with the property that $\forall s > s_*$ there exist c_*, K_1, K_2, K_3, s.t. for $c > c_*$

$$\|\psi_0\| \leq \frac{K_1}{c^\alpha} \Longrightarrow \|\psi(t)\| \leq \frac{2K_1}{c^\alpha} \ , \quad \text{for } |t| \leq K_2 c^r \ .$$

For the same times one has

$$\sum_k k^{2s} \left| |\psi_k(t)|^2 - |\psi_k(0)|^2 \right| \leq \frac{K_3}{c^{4\alpha}} \ .$$

- Small initial data
- Longer time description, but the limit equation is not identified.

The end

THANK YOU