Bilinear discretization of quadratic vector fields

Yuri B. Suris

(Technische Universität Berlin)

Oberwolfach Workshop "Geometric Integration", MFO, 24.03.2011

Kahan's discretization

- W. Kahan. *Unconventional numerical methods for trajectory calculations* (Unpublished lecture notes, 1993).

$$\dot{x} = Q(x) + Bx + c \quad \rightsquigarrow \quad (\widetilde{x} - x)/\epsilon = Q(x, \widetilde{x}) + B(x + \widetilde{x})/2 + c,$$

where $B \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}^n$, each component of $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a *quadratic* form, and $Q(x, \widetilde{x}) = (Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x}))/2$ is the corresponding symmetric *bilinear* function. Thus,

$$\dot{x}_k \rightsquigarrow (\widetilde{x}_k - x_k)/\epsilon, \quad x_k^2 \rightsquigarrow x_k \widetilde{x}_k, \quad x_j x_k \rightsquigarrow (x_j \widetilde{x}_k + \widetilde{x}_j x_k)/2.$$

Note: equations for $\widetilde{x}$ always linear, the map $\widetilde{x} = f(x, \epsilon)$ is always reversible and birational,

$$f^{-1}(x, \epsilon) = f(x, -\epsilon).$$

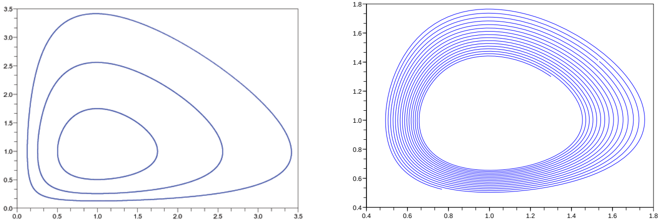
Illustration: Lotka-Volterra system

Kahan's integrator for the Lotka-Volterra system:

$$\begin{cases} \dot{x} = x(1-y), \\ \dot{y} = y(x-1), \end{cases} \quad \rightsquigarrow \quad \begin{cases} \widetilde{x} - x = \epsilon(\widetilde{x} + x) - \epsilon(\widetilde{x}y + x\widetilde{y}), \\ \widetilde{y} - y = \epsilon(\widetilde{x}y + x\widetilde{y}) - \epsilon(\widetilde{y} + y). \end{cases}$$

Explicitly:

$$\begin{cases} \widetilde{x} = x \, \dfrac{(1+\epsilon)^2 - \epsilon(1+\epsilon)x - \epsilon(1-\epsilon)y}{1-\epsilon^2 - \epsilon(1-\epsilon)x + \epsilon(1+\epsilon)y}, \\[2ex] \widetilde{y} = y \, \dfrac{(1-\epsilon)^2 + \epsilon(1+\epsilon)x + \epsilon(1-\epsilon)y}{1-\epsilon^2 - \epsilon(1-\epsilon)x + \epsilon(1+\epsilon)y}. \end{cases}$$



Left: three orbits of Kahan's discretization with $\epsilon = 0.1$,
right: one orbit of the explicit Euler with $\epsilon = 0.01$.

- J.M. Sanz-Serna. *An unconventional symplectic integrator of W.Kahan.* Applied Numer. Math. **16** (1994) 245–250.

A sort of an explanation of a non-spiralling behavior: Kahan's integrator for the Lotka-Volterra system is symplectic (Poisson).

The problem of integrable discretization. Hamiltonian approach (Birkhäuser, 2003)

Consider a completely integrable flow

$$\dot{x} = f(x) = \{H, x\} \qquad (1)$$

with a Hamilton function H on a Poisson manifold $\mathcal{P}$ with a Poisson bracket $\{\cdot, \cdot\}$. Thus, the flow (1) possesses many functionally independent integrals $I_k(x)$ in involution.

The *problem of integrable discretization*: find a family of diffeomorphisms $\mathcal{P} \to \mathcal{P}$,

$$\widetilde{x} = \Phi(x; \epsilon), \qquad (2)$$

depending smoothly on a small parameter $\epsilon > 0$, with the following properties:

1. The maps (2) *approximate* the flow (1):

$$\Phi(x;\epsilon) = x + \epsilon f(x) + O(\epsilon^2).$$

2. The maps (2) are *Poisson* w. r. t. the bracket $\{\cdot,\cdot\}$ or some its deformation $\{\cdot,\cdot\}_\epsilon = \{\cdot,\cdot\} + O(\epsilon)$.
3. The maps (2) are *integrable*, i.e. possess the necessary number of independent integrals in involution, $I_k(x;\epsilon) = I_k(x) + O(\epsilon)$.

While integrable lattice systems (like Toda or Volterra lattices) can be discretized in a systematic way (based, e.g., on the r-matrix structure), there is no systematic way to obtain decent integrable discretizations for integrable systems of classical mechanics.

Missing in the book: Hirota-Kimura discretizations

- R.Hirota, K.Kimura. *Discretization of the Euler top.* J. Phys. Soc. Japan **69** (2000) 627–630,
- K.Kimura, R.Hirota. *Discretization of the Lagrange top.* J. Phys. Soc. Japan **69** (2000) 3193–3199.

Reasons for this omission: discretization of the Euler top seemed to be an isolated curiosity; discretization of the Lagrange top seemed to be completely incomprehensible, if not even wrong.

Renewed interest stimulated by a talk by T. Ratiu at the Oberwolfach Workshop "Geometric Integration", March 2006, who claimed that HK-type discretizations for the Clebsch system and for the Kovalevsky top are also integrable.

Hirota-Kimura's discrete time Euler top

$$\left\{\begin{array}{l} \dot{x}_1 = \alpha_1 x_2 x_3, \\ \dot{x}_2 = \alpha_2 x_3 x_1, \\ \dot{x}_3 = \alpha_3 x_1 x_2, \end{array}\right. \quad \rightsquigarrow \quad \left\{\begin{array}{l} \widetilde{x}_1 - x_1 = \epsilon\alpha_1(\widetilde{x}_2 x_3 + x_2\widetilde{x}_3), \\ \widetilde{x}_2 - x_2 = \epsilon\alpha_2(\widetilde{x}_3 x_1 + x_3\widetilde{x}_1), \\ \widetilde{x}_3 - x_3 = \epsilon\alpha_3(\widetilde{x}_1 x_2 + x_1\widetilde{x}_2). \end{array}\right.$$

Features:

- Equations are linear w.r.t. $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)^{\mathrm{T}}$:

$$A(x,\epsilon)\widetilde{x} = x, \qquad A(x,\epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix},$$

  result in an *explicit* (rational) map, which is *reversible* (therefore birational):

$$\widetilde{x} = f(x,\epsilon) = A^{-1}(x,\epsilon)x, \quad f^{-1}(x,\epsilon) = f(x,-\epsilon).$$

- Explicit formulas rather messy:

$$\begin{cases} \widetilde{x}_1 = \dfrac{x_1 + 2\epsilon\alpha_1 x_2 x_3 + \epsilon^2 x_1(-\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_2 = \dfrac{x_2 + 2\epsilon\alpha_2 x_3 x_1 + \epsilon^2 x_2(\alpha_2\alpha_3 x_1^2 - \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \\ \widetilde{x}_3 = \dfrac{x_3 + 2\epsilon\alpha_3 x_1 x_2 + \epsilon^2 x_3(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 - \alpha_1\alpha_2 x_3^2)}{\Delta(x,\epsilon)}, \end{cases}$$

where $\Delta(x,\epsilon) = \det A(x,\epsilon)$

$$= 1 - \epsilon^2(\alpha_2\alpha_3 x_1^2 + \alpha_3\alpha_1 x_2^2 + \alpha_1\alpha_2 x_3^2) - 2\epsilon^3\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3.$$

(Try to see reversibility directly from these formulas!)

- Two independent integrals:

$$I_1(x,\epsilon)=\frac{1-\epsilon^2\alpha_2\alpha_3x_1^2}{1-\epsilon^2\alpha_3\alpha_1x_2^2},\qquad I_2(x,\epsilon)=\frac{1-\epsilon^2\alpha_3\alpha_1x_2^2}{1-\epsilon^2\alpha_1\alpha_2x_3^2}.$$

- Invariant volume form:

$$\omega=\frac{dx_1\wedge dx_2\wedge dx_3}{\phi(x)},\qquad \phi(x)=1-\epsilon^2\alpha_i\alpha_jx_k^2$$

  and bi-Hamiltonian structure found in:
  M. Petrera, Yu. Suris. *On the Hamiltonian structure of the Hirota-Kimura discretization of the Euler top.* Math. Nachr., 2010, **283**, 1654–1663, `arXiv:0707.4382 [math-ph]`.

Hirota-Kimura's discrete time Lagrange top

Equations of motion of the Lagrange top:

$$\begin{aligned}
\dot{m}_1 &= (\alpha-1)m_2m_3+\gamma p_2,\\
\dot{m}_2 &= (1-\alpha)m_1m_3-\gamma p_1,\\
\dot{m}_3 &= 0,\\
\dot{p}_1 &= \alpha p_2m_3-p_3m_2,\\
\dot{p}_2 &= p_3m_1-\alpha p_1m_3,\\
\dot{p}_3 &= p_1m_2-p_2m_1.
\end{aligned}$$

It is Hamiltonian w.r.t. Lie-Poisson bracket of $e(3)$, has four functionally independent integrals in involution: two Casimir functions,

$$C_1=p_1^2+p_2^2+p_3^2,\quad C_2=m_1p_1+m_2p_2+m_3p_3,$$

the Hamilton function, and the (trivial) "fourth integral",

$$H_1=\frac{1}{2}(m_1^2+m_2^2+\alpha m_3^2)+\gamma p_3,\quad H_2=m_3.$$

Discretization:

$$
\begin{aligned}
\widetilde{m}_1 - m_1 &= \epsilon(\alpha-1)(\widetilde{m}_2 m_3 + m_2\widetilde{m}_3) + \epsilon\gamma(p_2+\widetilde{p}_2),\\
\widetilde{m}_2 - m_2 &= \epsilon(1-\alpha)(\widetilde{m}_1 m_3 + m_1\widetilde{m}_3) - \epsilon\gamma(p_1+\widetilde{p}_1),\\
\widetilde{m}_3 - m_3 &= 0,\\
\widetilde{p}_1 - p_1 &= \epsilon\alpha(p_2\widetilde{m}_3 + \widetilde{p}_2 m_3) - \epsilon(p_3\widetilde{m}_2 + \widetilde{p}_3 m_2),\\
\widetilde{p}_2 - p_2 &= \epsilon(p_3\widetilde{m}_1 + \widetilde{p}_3 m_1) - \epsilon\alpha(p_1\widetilde{m}_3 + \widetilde{p}_1 m_3),\\
\widetilde{p}_3 - p_3 &= \epsilon(p_1\widetilde{m}_2 + \widetilde{p}_1 m_2 - p_2\widetilde{m}_1 - \widetilde{p}_2 m_1).
\end{aligned}
$$

As usual, this gives an explicit birational map $(\widetilde{m},\widetilde{p}) = f(m,p,\epsilon)$. The trivial conserved quantity $m_3 = \mathrm{const}$. Quite nontrivial to find any further conserved quantity!

Hirota-Kimura's "method" for finding integrals

Consider the expression $A = m_1^2 + m_2^2 - Bp_3 - Cp_3^2$, and determine A, B, C by requiring that they are conserved quantities. For this aim, solve the system of three equations for these three unknowns:

$$\begin{cases} A + B\widetilde{p}_3 + C\widetilde{p}_3^2 = \widetilde{m}_1^2 + \widetilde{m}_2^2, \\ A + Bp_3 + Cp_3^2 = m_1^2 + m_2^2, \\ A + B\underset{\sim}{p}_3 + C\underset{\sim}{p}_3^2 = \underset{\sim}{m}_1^2 + \underset{\sim}{m}_2^2 \end{cases}$$

with $(\widetilde{m}, \widetilde{p}) = f(m, p, \epsilon)$ and $(\underset{\sim}{m}, \underset{\sim}{p}) = f^{-1}(m, p, \epsilon)$. Then check that $A, B, C = A, B, C(m, p, \epsilon)$ are conserved quantities, indeed. Proceed similarly to determine the conserved quantities $D, \ldots, M$ from

$$D = m_1 p_1 + m_2 p_2 - Ep_3 - Fp_3^2, \qquad K = p_1^2 + p_2^2 - Lp_3 - Mp_3^2.$$

Does this make any sense for you???

Nevertheless, Hirota-Kimura's "method" turns out to be valid in this case and also for remarkably many other Hirota-Kimura type discretizations.

How should it be interpreted? Solve (symbolically) the system

$$(A + Bp_3 + Cp_3^2) \circ f^i(m, p, \epsilon) = (m_1^2 + m_2^2) \circ f^i(m, p, \epsilon)$$

with $i = -1, 0, 1$. Verify that $A = A \circ f,\ B = B \circ f,\ C = C \circ f$. Alternatively, solve another copy of the above system with $i = 0, 1, 2$, and check that the solutions coincide. But then this system should be satisfied for all $i \in \mathbb{Z}$. In other words, for any $(m, p) \in \mathbb{R}^6$, *certain linear combination*

$$A + Bp_3 + Cp_3^2 - (m_1^2 + m_2^2)$$

vanishes along the orbit of (m, p) *under the map* f.
This is a very special feature of both the map f and the set of functions $(1, p_3, p_3^2, m_1^2 + m_2^2)$. Also the sets of functions

$$(1, p_3, p_3^2, m_1 p_1 + m_2 p_2), \quad (1, p_3, p_3^2, p_1^2 + p_2^2)$$

have this property. It is formalized in the following definition.

Hirota-Kimura bases

Definition. *For a given birational map $f : \mathbb{R}^n \to \mathbb{R}^n$, a set of functions $\Phi = (\varphi_1, \ldots, \varphi_l)$, linearly independent over $\mathbb{R}$, is called a* **HK-basis**, *if for every $x_0 \in \mathbb{R}^n$ there exists a vector $c = (c_1, \ldots, c_l) \neq 0$ such that*

$$c_1\varphi_1(f^i(x_0)) + \ldots + c_l\varphi_l(f^i(x_0)) = 0 \quad \forall i \in \mathbb{Z}.$$

For a given $x_0 \in \mathbb{R}^n$, the set of all vectors $c \in \mathbb{R}^l$ with this property will be denoted by $K_\Phi(x_0)$ and called the null-space of the basis Φ (at the point x_0). This set clearly is a vector space.

Note: we cannot claim that $h = c_1\varphi_1 + ... + c_l\varphi_l$ is an integral of motion, since vectors $c \in K_\Phi(x_0)$ vary from one initial point x_0 to another.

However: existence of a HK-basis Φ with $\dim K_\Phi(x_0) = d$ confines the orbits of f to $(n - d)$-dimensional invariant sets.

From HK-bases to integrals

Proposition. *If* Φ *is a HK-basis for a map* f*, then* $K_\Phi(f(x_0)) = K_\Phi(x_0)$.

Thus, the d-dimensional null-space $K_\Phi(x_0)$ is a $Gr(d, l)$-valued integral. Its Plücker coordinates are scalar integrals.

Especially simple is the situation when the null-space of a HK-basis has dimension $d = 1$.

Corollary. *Let* Φ *be a HK-basis for* f *with* $\dim K_\Phi(x_0) = 1$ *for all* $x_0 \in \mathbb{R}^n$*. Let* $K_\Phi(x_0) = [c_1(x_0) : \ldots : c_l(x_0)] \in \mathbb{RP}^{l-1}$*. Then the functions* c_j/c_k *are integrals of motion for* f*.*

In other words, normalizing $c_l(x_0) = 1$ (say), we find that all other c_j ($j = 1, \ldots, l-1$) are integrals of motion. It is not clear whether one can say something general about the number of functionally independent integrals among them. It varies in examples (sometimes just $= 1$ and sometimes > 1).

Hirota-Kimura bases for the discrete Lagrange top

Thus, results by Hirota and Kimura in the Lagrange top case can be put as follows:

Theorem. *The three sets of functions,*

$$\begin{aligned}\Phi_1 &= (m_1^2+m_2^2,\ p_3^2,\ p_3,\ 1),\\ \Phi_2 &= (m_1p_1+m_2p_2,\ p_3^2,\ p_3,\ 1),\\ \Phi_3 &= (p_1^2+p_2^2,\ p_3^2,\ p_3,\ 1),\end{aligned}$$

are HK-bases for the discrete time Lagrange top with one-dimensional null-spaces.

It follows that any orbit lies on a two-dimensional surface in $\mathbb{R}^6$ which is intersection of three quadrics and a hyperplane $m_3 = \mathrm{const}$.

An impression about the complexity of the integrals thus found can be given by this: $K_{\Phi_1}(x) = [c_0 : c_1 : c_2 : -1]$, with

$$c_0 = \frac{m_1^2 + m_2^2 + 2\gamma p_3 + \epsilon^2 c_0^{(4)} + \epsilon^4 c_0^{(6)} + \epsilon^6 c_0^{(8)} + \epsilon^8 c_0^{(10)}}{\Delta_1 \Delta_2},$$

(and similar expressions for c_1, c_2), where

$$\begin{aligned} \Delta_1 &= 1 + \epsilon^2 \alpha(1-\alpha) m_3^2 - \epsilon^2 \gamma p_3, \\ \Delta_2 &= 1 + \epsilon^2 \Delta_2^{(2)} + \epsilon^4 \Delta_2^{(4)} + \epsilon^6 \Delta_2^{(6)}; \end{aligned}$$

coefficients $\Delta^{(q)}$ and $c_k^{(q)}$ are polynomials of degree q in the phase variables.

A simple integral (unnoticed by Hirota and Kimura) is given by:
Theorem. *The set*

$$\Gamma = (\widetilde{m}_1 p_1 - m_1 \widetilde{p}_1,\ \widetilde{m}_2 p_2 - m_2 \widetilde{p}_2,\ \widetilde{m}_3 p_3 - m_3 \widetilde{p}_3)$$

is a HK-basis for the discrete time Lagrange top with one-dimensional null-space $K_\Gamma(x) = [1 : 1 : I]$*,*

$$I = \frac{(2\alpha - 1) + \epsilon^2(\alpha - 1)(m_1^2 + m_2^2) + \epsilon^2\gamma(m_1 p_1 + m_2 p_2)/m_3}{1 + \epsilon^2\alpha(1-\alpha)m_3^2 - \epsilon^2\gamma p_3}.$$

Another result which was unknown to Hirota and Kimura reads:
Theorem. *The discrete time Lagrange top possesses an invariant volume form:*

$$f^*\omega = \omega, \quad \omega = \frac{dm_1 \wedge dm_2 \wedge dm_3 \wedge dp_1 \wedge dp_2 \wedge dp_3}{\Delta_2(m,p)}.$$

Further examples of integrable HK-discretizations

Work in progress with A. Pfadler, M. Petrera. An overview given in `arXiv:1008.1040 [nlin.SI]` (to appear in *Regular and Chaotic Dyn.*). An (incomplete) list of examples:

- Three-wave interaction system.
- Periodic Volterra chain of period $N = 3, 4$:

$$\dot{x}_k = x_k(x_{k+1} - x_{k-1}), \quad k \in \mathbb{Z}/N\mathbb{Z}$$

- Dressing chain with $N = 3$:

$$\dot{x}_k + \dot{x}_{k+1} = x_{k+1}^2 - x_k^2 + \alpha_{k+1} - \alpha_k, \quad k \in \mathbb{Z}/N\mathbb{Z}, \quad N \text{ odd.}$$

- System of two interacting Euler tops.
- Kirchhof and Clebsch cases of the rigid body motion in an ideal fluid.

Clebsch case

Clebsch case of the motion of a rigid body in an ideal fluid:

$$\begin{aligned}
\dot{m}_1 &= (\omega_3 - \omega_2) p_2 p_3, \\
\dot{m}_2 &= (\omega_1 - \omega_3) p_3 p_1, \\
\dot{m}_3 &= (\omega_2 - \omega_1) p_1 p_2, \\
\dot{p}_1 &= m_3 p_2 - m_2 p_3, \\
\dot{p}_2 &= m_1 p_3 - m_3 p_1, \\
\dot{p}_3 &= m_2 p_1 - m_1 p_2.
\end{aligned}$$

It is Hamiltonian w.r.t. Lie-Poisson bracket of $e(3)$, has four functionally independent integrals in involution:

$$I_i = p_i^2 + \frac{m_j^2}{\omega_k - \omega_i} + \frac{m_k^2}{\omega_j - \omega_i}, \quad (i,j,k) = c.p.(1,2,3),$$

and $H_4 = m_1 p_1 + m_2 p_2 + m_3 p_3$.

Hirota-Kimura discretization of the Clebsch system

Hirota-Kimura-type discretization (proposed by T. Ratiu on Oberwolfach Meeting "Geometric Integration", March 2006):

$$
\begin{aligned}
\widetilde{m}_1 - m_1 &= \epsilon(\omega_3 - \omega_2)(\widetilde{p}_2 p_3 + p_2 \widetilde{p}_3),\\
\widetilde{m}_2 - m_2 &= \epsilon(\omega_1 - \omega_3)(\widetilde{p}_3 p_1 + p_3 \widetilde{p}_1),\\
\widetilde{m}_3 - m_3 &= \epsilon(\omega_2 - \omega_1)(\widetilde{p}_1 p_2 + p_1 \widetilde{p}_2),\\
\widetilde{p}_1 - p_1 &= \epsilon(\widetilde{m}_3 p_2 + m_3 \widetilde{p}_2) - \epsilon(\widetilde{m}_2 p_3 + m_2 \widetilde{p}_3),\\
\widetilde{p}_2 - p_2 &= \epsilon(\widetilde{m}_1 p_3 + m_1 \widetilde{p}_3) - \epsilon(\widetilde{m}_3 p_1 + m_3 \widetilde{p}_1),\\
\widetilde{p}_3 - p_3 &= \epsilon(\widetilde{m}_2 p_1 + m_2 \widetilde{p}_1) - \epsilon(\widetilde{m}_1 p_2 + m_1 \widetilde{p}_2).
\end{aligned}
$$

What follows is based on: M. Petrera, A. Pfadler, Yu. Suris. *On integrability of Hirota-Kimura type discretizations. Experimental study of the discrete Clebsch system.* Experimental Math., 2009, **18**, 223–247, `arXiv:0808.3345 [nlin.SI]`

A birational map

$$\begin{pmatrix} \widetilde{m} \\ \widetilde{p} \end{pmatrix} = f(m, p, \epsilon) = M^{-1}(m, p, \epsilon) \begin{pmatrix} m \\ p \end{pmatrix},$$

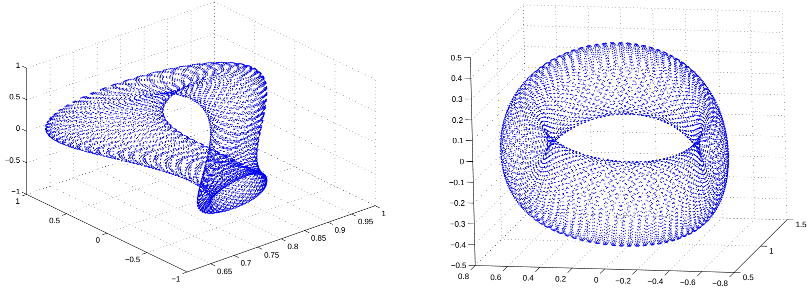
$$M(m, p, \epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & \epsilon\omega_{23}p_3 & \epsilon\omega_{23}p_2 \\ 0 & 1 & 0 & \epsilon\omega_{31}p_3 & 0 & \epsilon\omega_{31}p_1 \\ 0 & 0 & 1 & \epsilon\omega_{12}p_2 & \epsilon\omega_{12}p_1 & 0 \\ 0 & \epsilon p_3 & -\epsilon p_2 & 1 & -\epsilon m_3 & \epsilon m_2 \\ -\epsilon p_3 & 0 & \epsilon p_1 & \epsilon m_3 & 1 & -\epsilon m_1 \\ \epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1 \end{pmatrix},$$

with $\omega_{ij} = \omega_i - \omega_j$. The usual reversibility:

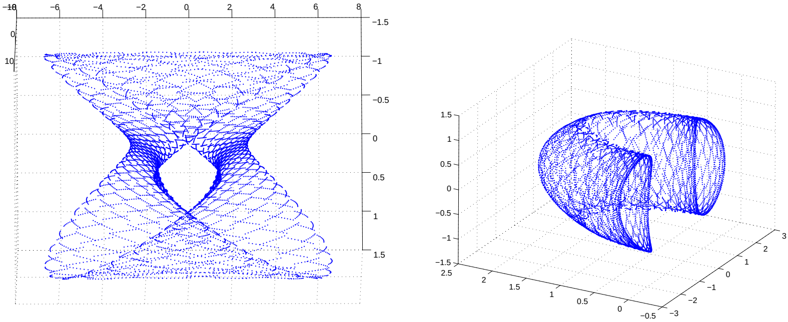
$$f^{-1}(m, p, \epsilon) = f(m, p, -\epsilon).$$

Numerators and denominators of components of $\widetilde{m}, \widetilde{p}$ are polynomials of degree 6, the numerators of $\widetilde{p}_i$ consist of 31 monomials, the numerators of $\widetilde{m}_i$ consist of 41 monomials, the common denominator consists of 28 monomials.

Phase portraits



An orbit of the discrete Clebsch system with $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = 0.3$ and $\epsilon = 1$; projections to (m_1, m_2, m_3) and to (p_1, p_2, p_3); initial point $(m_0, p_0) = (1, 1, 1, 1, 1, 1)$.



An orbit of the discrete Clebsch system with $\omega_1 = 1$, $\omega_2 = 0.2$, $\omega_3 = 30$ and $\epsilon = 1$; projections to (m_1, m_2, m_3) and to (p_1, p_2, p_3); initial point $(m_0, p_0) = (1, 1, 1, 1, 1, 1)$.

Results for the discrete Clebsch system

Theorem. *a) The set of functions*

$$\Phi = (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1 p_1, m_2 p_2, m_3 p_3, 1)$$

is a HK-basis for f*, with* $\dim K_\Phi(m,p) = 4$*. Thus, any orbit of* f *lies on an intersection of four quadrics in* $\mathbb{R}^6$*.*
b) The following four sets of functions are HK-bases for f *with one-dimensional null-spaces:*

$$\begin{aligned} \Phi_0 &= (p_1^2, p_2^2, p_3^2, 1), \\ \Phi_1 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_1 p_1), \\ \Phi_2 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_2 p_2), \\ \Phi_3 &= (p_1^2, p_2^2, p_3^2, m_1^2, m_2^2, m_3^2, m_3 p_3). \end{aligned}$$

There holds: $K_\Phi = K_{\Phi_0} \oplus K_{\Phi_1} \oplus K_{\Phi_2} \oplus K_{\Phi_3}$*.*

HK-basis Φ_0

Theorem. *At each point* $(m,p)\in\mathbb{R}^6$ *there holds:*

$$K_{\Phi_0}(m,p) = \left[\,\frac{1}{J_0}+\epsilon^2\omega_1 : \frac{1}{J_0}+\epsilon^2\omega_2 : \frac{1}{J_0}+\epsilon^2\omega_3 : -1\,\right],$$

where

$$J_0(m,p,\epsilon) = \frac{p_1^2+p_2^2+p_3^2}{1-\epsilon^2(\omega_1 p_1^2+\omega_2 p_2^2+\omega_3 p_3^2)}\,.$$

This function is an integral of motion of the map f*.*

This is the only “simple” integral of f!

HK-bases Φ_1, Φ_2, Φ_3

Theorem. *At each point $(m,p) \in \mathbb{R}^6$ there holds:*

$$\begin{aligned} K_{\Phi_1}(m,p) &= [\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_5 : \alpha_6 : -1], \\ K_{\Phi_2}(m,p) &= [\beta_1 : \beta_2 : \beta_3 : \beta_4 : \beta_5 : \beta_6 : -1], \\ K_{\Phi_3}(m,p) &= [\gamma_1 : \gamma_2 : \gamma_3 : \gamma_4 : \gamma_5 : \gamma_6 : -1], \end{aligned}$$

where α_j,β_j, and γ_j are rational functions of (m,p), even with respect to ϵ. They are integrals of motion of the map f. For $j = 1,2,3$ they are of the form

$$h = \frac{h^{(2)} + \epsilon^2 h^{(4)} + \epsilon^4 h^{(6)} + \epsilon^6 h^{(8)} + \epsilon^8 h^{(10)} + \epsilon^{10} h^{(12)}}{2\epsilon^2(p_1^2 + p_2^2 + p_3^2)\Delta},$$

$$\Delta = m_1 p_1 + m_2 p_2 + m_3 p_3 + \epsilon^2 \Delta^{(4)} + \epsilon^4 \Delta^{(6)} + \epsilon^6 \Delta^{(8)},$$

where h stands for any of the functions $\alpha_j, \beta_j, \gamma_j$, $j = 1,2,3$, and the corresponding $h^{(2q)}$ and $\Delta^{(2q)}$ are homogeneous polynomials in phase variables of degree $2q$. For instance,

HK-bases Φ_1, Φ_2, Φ_3 (continued)

$$
\begin{array}{lll}
\alpha_1^{(2)} = H_3 - I_1, & \alpha_2^{(2)} = -I_1, & \alpha_3^{(2)} = -I_1, \\
\beta_1^{(2)} = -I_2, & \beta_2^{(2)} = H_3 - I_2, & \beta_3^{(2)} = -I_2, \\
\gamma_1^{(2)} = -I_3, & \gamma_2^{(2)} = -I_3, & \gamma_3^{(2)} = H_3 - I_3,
\end{array}
$$

where $H_3 = p_1^2 + p_2^2 + p_3^2$. The four integrals J_0, α_1, β_1 and γ_1 are functionally independent.

Complexity issues

The claims of the last two theorems refer to the solutions of the following systems:

$$(c_1 p_1^2 + c_2 p_2^2 + c_3 p_3^2) \circ f^i = 1,$$

$$(\alpha_1 p_1^2 + \alpha_2 p_2^2 + \alpha_3 p_3^2 + \alpha_4 m_1^2 + \alpha_5 m_2^2 + \alpha_6 m_3^2) \circ f^i = m_1 p_1 \circ f^i,$$

$$(\beta_1 p_1^2 + \beta_2 p_2^2 + \beta_3 p_3^2 + \beta_4 m_1^2 + \beta_5 m_2^2 + \beta_6 m_3^2) \circ f^i = m_2 p_2 \circ f^i,$$

$$(\gamma_1 p_1^2 + \gamma_2 p_2^2 + \gamma_3 p_3^2 + \gamma_4 m_1^2 + \gamma_5 m_2^2 + \gamma_6 m_3^2) \circ f^i = m_3 p_3 \circ f^i.$$

The first one has to be solved for one non-symmetric range of $I - 1 = 3$ values of i, or for two different such ranges. The last three systems have to be solved for a non-symmetric range of $I - 1 = 6$ values of i. This can be done numerically (in rational arithmetic) without any difficulties, but becomes (nearly) impossible for a symbolic computation, due to complexity of f^2.

Complexity of f^2

Degrees of numerators and denominators of f^2:

	deg	$\deg_{p_1}$	$\deg_{p_2}$	$\deg_{p_3}$	$\deg_{m_1}$	$\deg_{m_2}$	$\deg_{m_3}$
Denom. of f^2	27	24	24	24	12	12	12
Num. of $p_1 \circ f^2$	27	25	24	24	12	12	12
Num. of $p_2 \circ f^2$	27	24	25	24	12	12	12
Num. of $p_3 \circ f^2$	27	24	24	25	12	12	12
Num. of $m_1 \circ f^2$	33	28	28	28	15	14	14
Num. of $m_2 \circ f^2$	33	28	28	28	14	15	14
Num. of $m_3 \circ f^2$	33	28	28	28	14	14	15

The numerator of the p_1-component of $f^2(m, p)$, as a polynomial of m_k, p_k, contains 64 056 monomials; as a polynomial of m_k, p_k, and ω_k, it contains 1 647 595 terms.

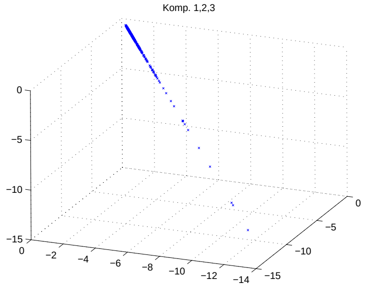
Need new ideas! The main one: find (observe numerically) linear relations between the components of $K_\Phi(x_0)$, and then use them to replace the dynamical relations.

Example

Plotting solutions (c_1, c_2, c_3) of the system

$$(c_1 p_1^2 + c_2 p_2^2 + c_3 p_3^2) \circ f^i = 1, \ i = 0, 1, 2$$

with varying initial data, we get:



Straight line $\rightsquigarrow$ two linear relations between (c_1, c_2, c_3)!

Summary for the Clebsch system

We established the integrability of the Hirota-Kimura discretization of the Clebsch system, in the sense of

- existence, for every initial point $(m, p) \in \mathbb{R}^6$, of a four-dimensional pencil of quadrics containing the orbit of this point;
- existence of four functionally independent integrals of motion (conserved quantities).

This remains true also for an arbitrary flow of the Clebsch system (with one "simple" and three very big integrals).

Our proofs are computer assisted. We did not find a general structure, which would provide us with less computational proofs and with more insight. In particular, nothing like a Lax representation has been found. Nothing is known about the existence of an invariant Poisson structure for these maps.

Conjecture

The previous discussion seems to support the following

Conjecture. *For any algebraically completely integrable system with a quadratic vector field, its Hirota-Kimura discretization remains algebraically completely integrable*

pushed forward in our paper in *"Exp. Math."*. However, at present we have a number of apparent counterexamples (it is extremely difficult to prove non-integrability), including the so called Zhukovsky-Volterra gyrostat. However, the HK discretization maintains integrability much more often than a mere coincidence would allow.

The full story still has to be clarified.