Discrete Mechanics and Optimal Control: Structure preserving integration for the optimal control of mechanical systems

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Motivation

A **Optimal control problem**
$\min_{x,u} J(x,u) \quad s.t.\ \dot{x}(t) = f(x(t), u(t)),\ x(0) = x^0$

- well established numerical methods
  indirect: solve necessary optimality conditions
  direct: discretize problem *in a clever way* and solve optimization problem

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B **Structure preserving integration**

approximate solution $x(t)$ of $\dot{x}(t) = f(x(t))$ via a discrete solution $\{x_k\}_{k=0}^N$ with e.g. $x_{k+1} = \Psi_h(x_k)$

- preserve conservation properties in discrete solution
- geometric integrators, symplectic integrators,
  here: variational integrators

Motivation

A+B **optimal control problem**

+ structure preserving integration

= ?

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?

- What is preservation in presence of control $u(t)$?
- Which properties can be derived from integration theory?
- What do we gain for optimal control problems?

Outline

- Optimal control: problem formulation
- Discrete Mechanics and Optimal Control
- Preservation properties and approximation error

briefly:

- Variational approach to multirate integration
- related works:
  Betsch, Bock, Bonnans, Diehl, Gerdts, Hager, Kobilarov, Leok, Leyendecker, Marsden, Ortiz, von Stryk, West, Chyba, Hairer, Vilmart, ...

Optimal Control Problem

$$\min_{x(\cdot),u(\cdot),(T)} J(x,u) = \int_0^T C(x(t),u(t))\,\mathrm{d}t + \Phi(x(T))$$

subject to

$$\begin{array}{lrcl}
\text{system equations} & \dot{x}(t) & = & f(x(t),u(t)), \\
\text{initial value} & x(0) & = & x_0, \\
\text{path constraints} & 0 & \leq & h(x(t),u(t)), \\
\text{final point constraint} & 0 & = & r(x(T)).
\end{array}$$

$$x \in \mathbb{R}^{n_x},\ u \in \mathbb{R}^{n_u},\ J : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R},\ C : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R},$$
$$\Phi : \mathbb{R}^{n_x} \to \mathbb{R},\ f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}, h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_h},\ r : \mathbb{R}^{n_x} \to \mathbb{R}^{n_r}.$$

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$\Phi : \mathbb{R}^{n_x} \to \mathbb{R}$, $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$, $h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_h}$, $r : \mathbb{R}^{n_x} \to \mathbb{R}^{n_r}$.

Necessary optimality conditions:
Pontryagin maximum principle

Necessary optimality conditions

Theorem (Pontryagin Maximum Principle)

Let (x^*, u^*) be an optimal solution. Then, there exists a function $\lambda : [0, T] \to \mathbb{R}^{n_x}$ and a vector $\alpha \in \mathbb{R}^{n_r}$ such that

$$\begin{aligned}
\mathcal{H}(x^*(t), u^*(t), \lambda(t)) &= \max_{u(t)\in U} \mathcal{H}(x(t), u(t), \lambda(t))\, \forall t \in [0, T], \\
\dot{x}^*(t) &= \nabla_\lambda \mathcal{H}(x^*(t), u^*(t), \lambda(t)),\ x^*(0) = x_0, \\
\dot{\lambda}(t) &= -\nabla_x \mathcal{H}(x^*(t), u^*(t), \lambda(t)), \\
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with the Hamiltonian
$\mathcal{H}(x(t), u(t), \lambda(t)) = -C(x(t), u(t)) + \lambda^T(t) \cdot f(x(t), u(t)).$

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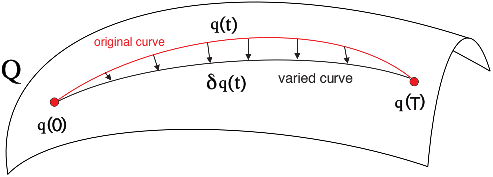
Mechanical system: variational formulation

- n-dimensional configuration manifold Q
- Lagrangian $L : TQ \to \mathbb{R}$, $L(q,\dot q) = K(q,\dot q) - V(q)$
- Lagrangian control force
  $f_L : TQ \times U \to T^*Q$



Lagrange-d'Alembert principle

$$\delta \int_0^T L(q(t),\dot q(t))\,\mathrm{d}t + \int_0^T f_L(q(t),\dot q(t),u(t)) \cdot \delta q(t)\,\mathrm{d}t = 0$$

$\Rightarrow$ Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot q} L(q(t),\dot q(t)) - \frac{\partial}{\partial q} L(q(t),\dot q(t)) = f_L(q(t),\dot q(t),u(t))$$

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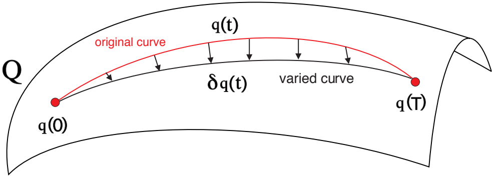

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$\Rightarrow \dot{x}(t) = f(x(t),u(t))$ with $x = (q,\dot{q})$

What are the relevant preservation properties?

Noether Theorem (no forcing)

Invariance of the Lagrangian under a group action of a Lie group G leads to the **preservation of momentum maps** along the trajectory.

- Lagrangian $L : TQ \to \mathbb{R}$, Lie group G, left action $\Psi_g : Q \to Q, g \in G$
- invariance of the Lagrangian: $L \circ \Psi_g^{TQ} = L$
- left action induces momentum map J with $J \circ F_L^t = J$ for all t with Lagrangian flow $F_L^t : TQ \to TQ$

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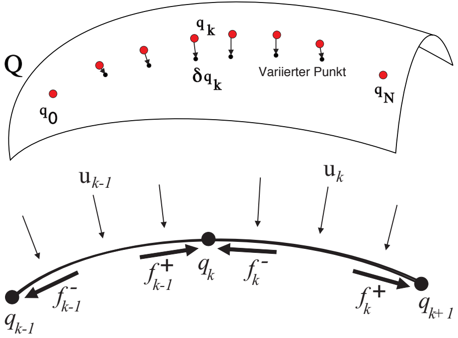
Symplecticity (no forcing)

Lagrangian flow is symplectic: $(F_L^t)^*(\Omega) = \Omega$.

Discrete variational principle

Replace

- state space TQ with $Q \times Q$
- curve $q(t) \in Q$ with sequence $q_d = \{q_k\}_{k=0}^{N}$
- curve $u(t) \in U$ with sequence $u_d = \{u_k\}_{k=0}^{N-1}$



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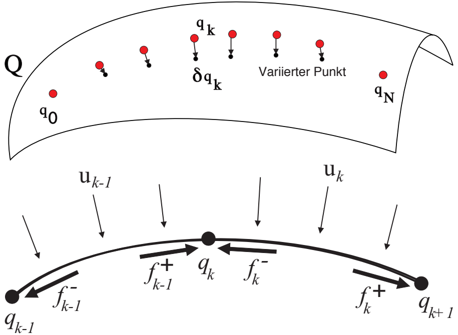

Approximate Lagrangian and virtual work:

- discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$

$$L_d(q_k, q_{k+1}) \approx \int_{kh}^{(k+1)h} L(q(t), \dot{q}(t)) \mathrm{d}t$$

- $f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} \approx \int_{kh}^{(k+1)h} f_L(q(t), \dot{q}(t), u(t)) \cdot \delta q(t)\, dt$

Discrete variational principle

- Discrete Lagrange-d'Alembert principle

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) + \sum_{k=0}^{N-1} \left[f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} \right] = 0$$

- Discrete Euler-Lagrange equations (DEL)

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) + f_k^- + f_{k-1}^+ = 0$$

- Variational integrator for forward simulation
  - ⊕ symplectic (preservation of symplectic form) ⇒ good long-time energy behavior — Hairer & Lubich (2004)
  - ⊕ preserves momentum maps, if L_d is invariant under group action — Marsden & West (2001)
  - order given by order of discrete Lagrangian and forces
- boundary conditions: continuous boundary conditions on TQ are transformed in discrete boundary conditions on $Q \times Q$ via the *discrete Legendre transformation*

Discrete optimal control problem

(DMOC: OB, Junge & Marsden (2008))

$$\min_{q_d, u_d, (h)} J_d(q_d, u_d) = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, u_k) + \Phi_d(q_{N-1}, q_N, u_{N-1})$$

s.t. DEL $\quad D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f_{k-1}^+ + f_k^- = 0$

initial conditions	$s_d(q_0, q_1, u_0, q^0, \dot{q}^0)$	$=$	0
path constraints	$h_d(q_k, q_{k+1}, u_k)$	$\geq$	0
final constraint	$r_d(q_{N-1}, q_N, u_{N-1}, q^T, \dot{q}^T)$	$=$	0

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$$\begin{array}{lrcl} \text{initial conditions} & s_d(q_0, q_1, u_0, q^0, \dot{q}^0) & = & 0 \\ \text{path constraints} & h_d(q_k, q_{k+1}, u_k) & \geq & 0 \\ \text{final constraint} & r_d(q_{N-1}, q_N, u_{N-1}, q^T, \dot{q}^T) & = & 0 \end{array}$$

$$\Downarrow$$

restricted optimization problem

$$\boxed{\min_\xi \varphi(\xi) \quad \text{s. t.} \quad a(\xi) = 0,\ b(\xi) \geq 0}$$

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$\Downarrow$

necessary optimality conditions

Karush-**K**uhn-**T**ucker: $\nabla\varphi(\xi) - \lambda^T \nabla a(\xi) - \mu^T \nabla b(\xi) = 0$
with Lagrange multipliers λ und μ.

Structure preservation in presence of forcing

Consequence from Noether:

- evolution of momentum map exactly determined by control forces
- preservation of momentum map if control forces act orthogonal to group action ($\langle f_L, \xi_Q \rangle = 0$)

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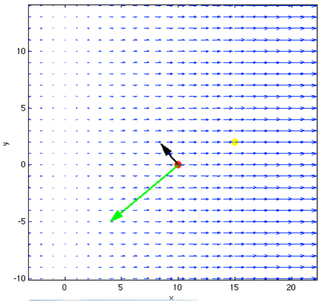
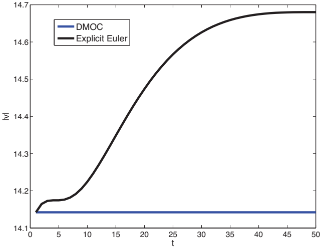
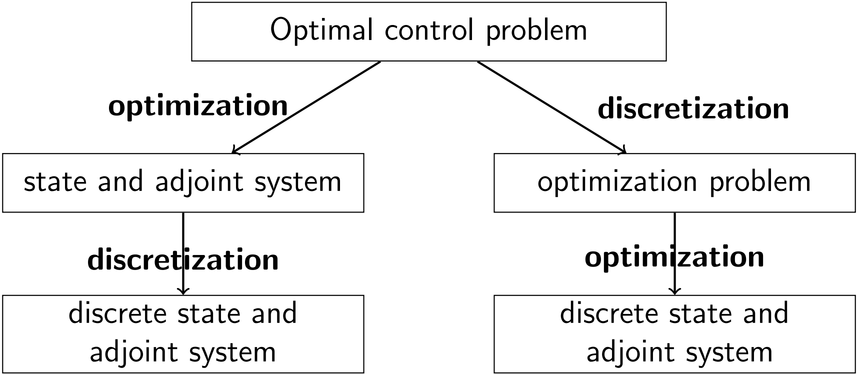
- preservation of momentum map if control forces act orthogonal to group action ($\langle f_L, \xi_Q \rangle = 0$)

Example: control force acts orthogonal to relative velocity $\Rightarrow$ exact preservation of velocity's absolute value





The adjoint system



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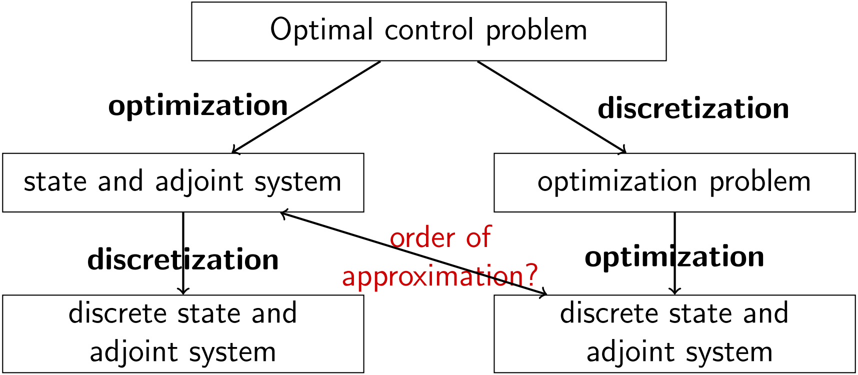
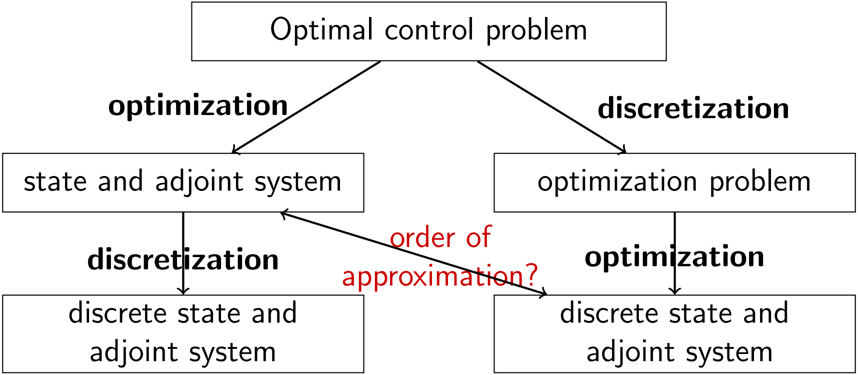


- RK methods: order of adjoint scheme is in general NOT the same as for state scheme

  [Hager 2000], [Bonnans, Laurent-Varin 2006]

The adjoint system



- RK methods: order of adjoint scheme is in general NOT the same as for state scheme

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- BUT: using the *right* scheme for the state system, state and adjoint scheme coincide (e.g. special class of VI)

Properties of DMOC: order of approximation

Proof strategy:

1 VI of order r given by $L_d = h\sum_{i=1}^{s} b_i L(Q_i, \dot{Q}_i)$

$$\Updownarrow$$

symplectic partitioned Runge-Kutta scheme of order r

[Suris 1990],[Marsden, West 2001]

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2. determine approximation order κ of adjoint system (as for RK methods [Hager 2000]):
   - necessary optimality conditions for continuous system (Pontryagin)
   - necessary optimality conditions for discrete system (KKT)

1 Hamiltonian formulation

- Lagrangian $(q, \dot{q})$ can be rewritten as Hamiltonian (q, p)
- partitioned Runge-Kutta scheme [Marsden, West 2001]

$$q_1 = q_0 + h \sum_{j=1}^{s} b_j^q \dot{Q}_j, \qquad p_1 = p_0 + h \sum_{j=1}^{s} b_j^p \dot{P}_j,$$

$$Q_i = q_0 + h \sum_{j=1}^{s} a_{ij}^q \dot{Q}_j, \qquad P_i = p_0 + h \sum_{j=1}^{s} a_{ij}^p \dot{P}_j$$

$$P_i = \frac{\partial L}{\partial \dot{q}}(Q_i, \dot{Q}_i), \qquad \dot{P}_i = \frac{\partial L}{\partial q}(Q_i, \dot{Q}_i) + f_L(Q_i, \dot{Q}_i, U_i)$$

$i = 1, \ldots, s$, with internal stages (Q_i, P_i), control samples $U_i = u_d(t_0 + c_i h)$, symplectic if $b_i^q a_{ij}^p + b_j^p a_{ji}^q = b_i^q b_j^p$, $b_i^q = b_i^p$.

2 Continuous adjoint system

- substitute optimal control $u(q(t), p(t), \lambda^q(t), \lambda^p(t))$ obtained by solving minimization problem
- two-point boundary problem

$$
\begin{aligned}
\dot{q}(t) &= \nu(q(t), p(t)), \quad q(0) = q^0, \\
\dot{p}(t) &= \eta(q(t), p(t), \lambda^q(t), \lambda^p(t)), \; p(0) = p^0, \\
\dot{\lambda}^q(t) &= \phi^q(q(t), p(t), \lambda^q(t), \lambda^p(t)), \; \lambda^q(T) = \Psi^q(q(T), p(T)), \\
\dot{\lambda}^p(t) &= \phi^p(q(t), p(t), \lambda^q(t), \lambda^p(t)), \; \lambda^p(T) = \Psi^p(q(T), p(T)).
\end{aligned}
$$

2 Discrete adjoint system (I)

$$
\begin{aligned}
q_{k+1} &= q_k + h \sum_{i=1}^{s} b_i \nu(Q_{ki}, P_{ki}), \quad q_0 = q^0, \\
p_{k+1} &= p_k + h \sum_{i=1}^{s} b_i \eta(Q_{ki}, P_{ki}, \chi_{ki}^q, \chi_{ki}^p), \quad p_0 = p^0, \\
\lambda_{k+1}^q &= \lambda_k^q + h \sum_{i=1}^{s} b_i \phi^q(Q_{ki}, P_{ki}, \chi_{ki}^q, \chi_{ki}^p), \quad \lambda_N^q = \Psi^q(q_N, p_N), \\
\lambda_{k+1}^p &= \lambda_k^p + h \sum_{i=1}^{s} b_i \phi^p(Q_{ki}, P_{ki}, \chi_{ki}^q, \chi_{ki}^p), \quad \lambda_N^p = \Psi^p(q_N, p_N),
\end{aligned}
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2 Discrete adjoint system (II)

$$
\begin{aligned}
Q_{ki} &= q_k + h\sum_{j=1}^{s} a_{ij}^{q}\nu(Q_{kj}, P_{kj}),\\
P_{ki} &= p_k + h\sum_{j=1}^{s} a_{ij}^{p}\eta(Q_{kj}, P_{kj}, \chi_{kj}^{q}, \chi_{kj}^{p}),\\
\chi_{ki}^{q} &= \lambda_k^{q} + h\sum_{j=1}^{s} \bar{a}_{ij}^{q}\phi^{q}(Q_{kj}, P_{kj}, \chi_{kj}^{q}, \chi_{kj}^{p}),\\
\chi_{ki}^{p} &= \lambda_k^{p} + h\sum_{j=1}^{s} \bar{a}_{ij}^{p}\phi^{p}(Q_{kj}, P_{kj}, \chi_{kj}^{q}, \chi_{kj}^{p}),\\
\bar{a}_{ij}^{q} &= \frac{b_i b_j - b_j a_{ji}^{q}}{b_i}, \quad \bar{a}_{ij}^{p} = \frac{b_i b_j - b_j a_{ji}^{p}}{b_i}.
\end{aligned}
$$

2 Order of approximation

Theorem (Order of approximation)

If the partitioned symplectic Runge-Kutta discretization of the state system is of order κ and $b_i > 0$ for each i, then the scheme for the adjoint system is again a partitioned symplectic Runge-Kutta scheme of the same order (in particular we obtain the same schemes for (q, p) and (λ^p, λ^q)).

Proof: With the symplecticity condition $a_{ij}^p = \frac{b_i b_j - b_j a_{ji}^q}{b_i}$ it holds

$$\begin{aligned} \bar{a}_{ij}^q &= a_{ij}^p, \\ \bar{a}_{ij}^p &= a_{ij}^q. \end{aligned}$$

$\square$

[OB, Junge, Marsden 2009]

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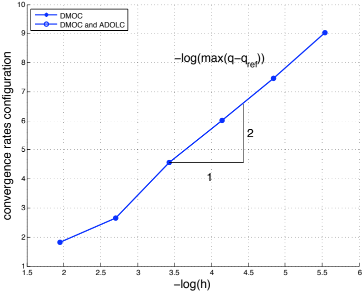
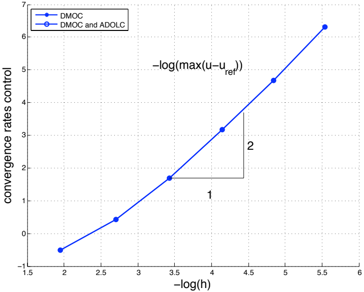
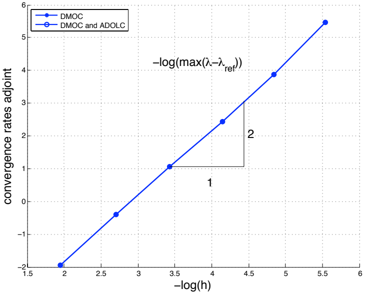
Properties of DMOC: convergence rates





- two-link pendulum optimally (minimal control effort) controlled from lower to upper equilibrium
- same convergence rates for configuration, control and adjoint



...to give answers to ?

A+B **optimal control problem**

+ structure preserving integration

=

- inherited from integration scheme (*var. formulation*): evolution of momentum maps due to symmetries determined by control forces
- gain for optimal control problems (*symplecticity*): order of adjoint system = order of state system

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extension to (holonomic) constrained systems

Optimal control of multi-body dynamics (DMOCC)

[Leyendecker, OB, Marsden, Ortiz 08]

- constraint manifold $\mathcal{C} = \{q \mid q \in Q, g(q) = 0\} \subset Q$
- discrete version of Lagrange-d'Alembert principle with augmented Lagrangian $\bar{L}(q, \dot{q}, \mu) = L(q, \dot{q}) - g^T(q) \cdot \mu$
- Forced constrained discrete EL equations

$$\begin{aligned} \left[D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) - hG^T(q_k) \cdot \mu_k + f_{k-1}^+ + f_k^-\right] &= 0 \\ g(q_{k+1}) &= 0 \end{aligned}$$

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$$
\begin{aligned}
P^T(q_k) \left[D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) - hG^T(q_k) \cdot \mu_k + f_{k-1}^+ + f_k^- \right] &= 0 \\
g(q_{k+1}) &= 0
\end{aligned}
$$

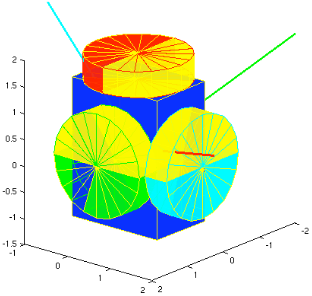
- dimension reduction via discrete nullspace matrix, relative reparametrization of configurations and forces

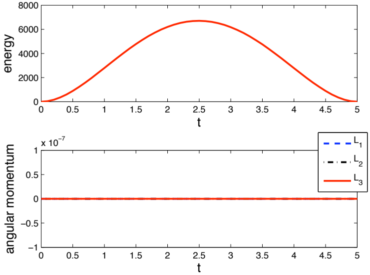
⊕ reduced scheme with minimal dimension
⊕ symplectic and momentum consistent
⊕ exact constraint fulfillment

Example (DMOCC): satellite reorientation

[LEYENDECKER, OB, MARSDEN, ORTIZ 08]

- satellite actuated via rotors attached at main body
- Goal: attitude control with minimal control effort
- preservation of total angular momentum





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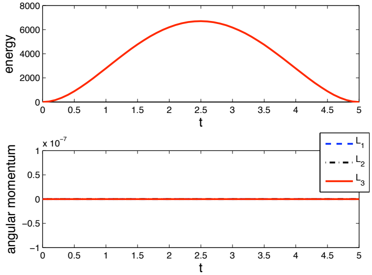


Variational multirate integration

(joint work with S. Leyendecker)

- mechanical systems with dynamics on different time scales
- Lagrangian

$$L(q,\dot{q}) = T(\dot{q}) - U(q), \quad U(q) = V(q) + W(q)$$

with $W(q) = \frac{1}{\varepsilon}\bar{W}(q)$, $\varepsilon \ll 1$.

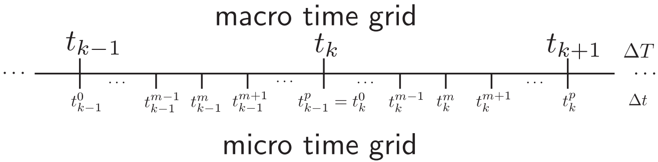
- assumption: separation in slow and fast variables $q = (q^s, q^f)$
- fast potential W only dependent on fast variables q^f
- Lagrangian

$$L(q,\dot{q}) = T(\dot{q}) - V(q) - W(q^f)$$

- aim: resolve slow and fast dynamics but with less function evaluations and smaller set of variables

Multirate: discrete approximation

- consider two time grids



macro grid $\{t_k = k\Delta T \mid k = 0, \dots, N\}$

micro grid $\{t_k^m = k\Delta T + m\Delta t \mid k = 0, \dots, N-1, m = 0, \dots, p\}$

- discrete slow variables $\{q_k^s\}_{k=0}^N$ with $q_k^s \approx q^s(t_k)$
- discrete fast variables $\{\{q_k^{f,m}\}_{m=0}^p\}_{k=0}^{N-1}$ with $q_k^{f,p} = q_{k+1}^{f,0}$ and $q_k^{f,m} \approx q^f(t_k^m)$

Discrete Lagrangian

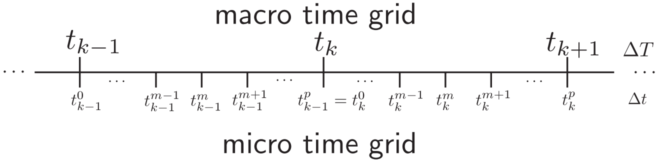
$$\bar{L}_d(q_k^s, q_{k+1}^s, q_k^{f,0}, \ldots, q_k^{f,p}) =$$

$$\sum_{m=0}^{p-1} L_d(q_k^s, q_{k+1}^s, q_k^{f,m}, q_k^{f,m+1})$$

$$= \sum_{m=0}^{p-1} \Big[T_d(q_k^s, q_{k+1}^s, q_k^{f,m}, q_k^{f,m+1})$$

$$- V_d(q_k^s, q_{k+1}^s, q_k^{f,m}, q_k^{f,m+1}) - W_d(q_k^{f,m}, q_k^{f,m+1}) \Big]$$

with $L_d = T_d - V_d - W_d$



Discrete variational principle

- discrete action

$$\mathfrak{S}_d\left(\{q_k^s\}_{k=0}^N, \{\{q_k^{f,m}\}_{m=0}^p\}_{k=0}^{N-1}\right) = \sum_{k=0}^{N-1}\sum_{m=0}^{p-1} L_d(q_k^s, q_{k+1}^s, q_k^{f,m}, q_k^{f,m+1})$$

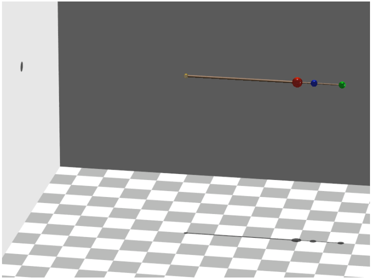
- stationary discrete action

$$\begin{aligned}\delta\mathfrak{S}_d &= \delta\sum_{k=0}^{N-1}\sum_{m=0}^{p-1} L_d(q_k^s, q_{k+1}^s, q_k^{f,m}, q_k^{f,m+1})\\ &= \sum_{k=0}^{N-1}\sum_{m=0}^{p-1} D_1 L_d\cdot\delta q_k^s + D_2 L_d\cdot\delta q_{k+1}^s\\ &+ \sum_{k=0}^{N-1}\sum_{m=0}^{p-1} D_3 L_d\cdot\delta q_k^{f,m} + D_4 L_d\cdot\delta q_k^{f,m+1} = 0\end{aligned}$$

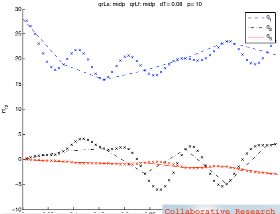
gives discrete Euler-Lagrange equations for fast and slow dynamics

Triple pendulum in 3D

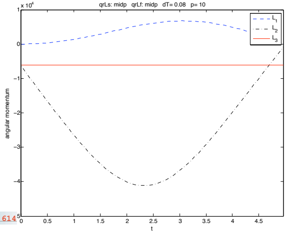
- different quadrature rules lead to different schemes
- invariance of discrete Lagrangian: preserved momentum maps
- discrete sympletic form preserved



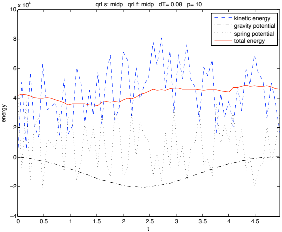
configuration



angular momentum



energy $(\Delta T = 0.08, p = 10)$



Conclusion and future directions

Conclusion

- fully discrete variational formulation for the optimal control of mechanical systems

  variational: structure preserving
  symplectic: approximation order of adjoint system
  = approximation order of state system

  $\rightarrow$ order of adjoint system also for extended DMOC versions (with holonomic constraints)

- variational multirate integration
  - unified variational approach for the simulation of mechanical systems with different time scales
  - structure preserving

  $\rightarrow$ analysis: comparison to existing methods, long-time behavior, efficiency

  $\rightarrow$ use for optimal control