Quasi-stroboscopic averaging: The autonomous case

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Geometric Numerical Integration, Oberwolfach 2011

A class of highly oscillatory Hamiltonian systems

We consider Hamiltonian systems with Hamiltonian function

$$H(x) = H^0(x) + K(x), \quad \text{with} \quad H^0(x) = \sum_{j=1}^{d} \omega_j I_j(x),$$

where we assume that

- $H^0(x) >> K(x)$,
- each $I_j(x)$ and $K(x)$ are smooth scalar functions ($x \in \mathbb{R}^{2n}$),
- $I_1(x), \ldots, I_d(x)$ are in involution,
- the t-flow $\Psi_t^{[j]}$ of each $I_j(x)$ is (2π)-periodic in t,
- the vector of frequencies $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$ is non-resonant, that is, $k \cdot \omega \neq 0$ if $0 \neq k \in \mathbb{Z}^d$.

Thus the t-flow $\Psi_{t\omega_1}^{[1]} \circ \cdots \circ \Psi_{t\omega_d}^{[d]}$ of $H^0(x)$ is quasi-periodic.

It includes as particular cases:

- Near-integrable systems in action-angle variables $x = (a, \theta) \in \mathbb{R}^n \times \mathbb{T}^n$ with Hamiltonian of the form

$$\sum_{j=1}^{n} \lambda_j \, a_j + \epsilon \, R(a, \theta).$$

- HOS conservative mechanical systems with Hamiltonian

$$H(p, q) = \frac{1}{2} p^T p + \frac{1}{2\epsilon^2} q^T \Lambda^2 q + U(q),$$

  where Λ is a symmetric matrix (with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$).

There exist $d \leq n$ and a non-resonant vector of frequencies $\omega \in \mathbb{R}^d$ such that the λ_j / ϵ are $\mathbb{Z}$-linear combinations of $\omega_1, \ldots, \omega_d$.

Example (A Fermi-Pasta-Ulam type problem (from HLW))

A Hamiltonian system with $n = 5$ ($q \in \mathbb{R}^5$, $p \in \mathbb{R}^5$),

$$\begin{aligned} H(p,q) &= \frac{1}{2}\sum_{j=1}^{5} p_j^2 + \frac{1}{2\epsilon^2}\sum_{j=1}^{5} \lambda_j^2 q_j^2 + U(q), \\ U(q) &= \frac{1}{2}q_1^2\left(1+\frac{1}{4}q_2^2\right) + \left(\frac{1}{20} + q_2 + q_3 + q_5 + \frac{5}{2}q_4\right)^4. \end{aligned}$$

where $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$, $\lambda_4 = 2$, $\lambda_5 = \sqrt{2}$.

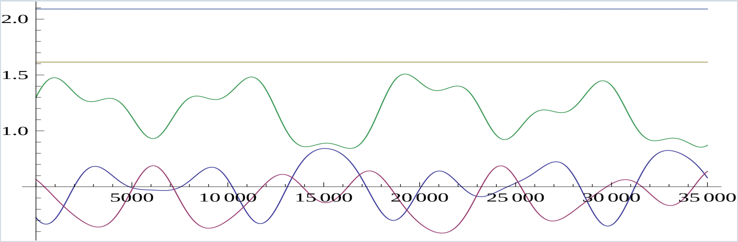
- Vector of non-resonant frequencies: $\omega = (\epsilon^{-1}, \sqrt{2}\,\epsilon^{-1}) \in \mathbb{R}^2$.
- $I_1(x) = \dfrac{\epsilon}{2}(p_2^2 + p_3^2 + p_4^2) + \dfrac{1}{2\epsilon}(q_2^2 + q_3^2 + 4q_4^2)$,
  $I_2(x) = \dfrac{\epsilon}{\sqrt{2}}p_5^2 + \dfrac{\sqrt{2}}{\epsilon}q_5^2$,
- $K(p,q) = \frac{1}{2}p_1^2 + U(q)$.

Example (continuation)

As in [HLW], we take $\epsilon = 1/70$, $p(0) = (-0.2, 0.6, 0.7, -0.9, 0.8)^T$, $q(0) = (1, 0.3\epsilon, 0.8\epsilon, -1.1\epsilon, 0.7\epsilon)^T$, and plot (versus t)

$$J_i(x) = \frac{1}{2}p_j^2 + \frac{\lambda_j^2}{2\epsilon^2}q_j^2, \quad j = 2, 3, 4, 5$$

$(J_5 = \omega_2 I_2)$ and $J_2 + J_3 + J_4 = \omega_1 I_1$



Formal first integrals and quasi-stroboscopic averaging

Under the precedent assumptions on $H(x) = \sum_{j=1}^{d} \omega_j \, I_j(x) + K(x)$,

there exists formal series $\tilde{I}_1(x), \ldots, \tilde{I}_d(x), \tilde{K}(x)$ such that

- $H(x) = \sum_{j=1}^{d} \omega_j \, \tilde{I}_j(x) + \tilde{K}(x)$,
- $H(x), \tilde{I}_1(x), \ldots, \tilde{I}_d(x), \tilde{K}(x)$ are in involution,
- the t-flow $\Phi_t^{[j]}$ of each $\tilde{I}_j(x)$ is (2π)-periodic in t,
- For any solution $x(t)$ of $\frac{d}{dt}x = J^{-1}\nabla H(x)$, $x(t) = \Phi_{t\omega_1}^{[1]} \circ \cdots \circ \Phi_{t\omega_d}^{[d]}(X(t))$, where $X(t)$ is the solution of the averaged system

$$\frac{d}{dt}X = J^{-1}\nabla \tilde{K}(X), \quad X(0) = x(0).$$

Consider the Fourier expansion of $K(\Psi_\theta(x)) = \sum_{k\in\mathbb{Z}^d} H_k(x) e^{i(k\cdot t)}$,

where $\Psi_\theta := \Psi^{[1]}_{\theta_1} \circ \cdots \circ \Psi^{[d]}_{\theta_d}$, each $\Psi^{[j]}_t$ being the t-flow of $I_j(x)$.

Explicit formulae for first integrals and averaged Hamiltonian

$$\begin{aligned}
\tilde{I}_j(x) &= I_j(x) + \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{k_j}{k\cdot\omega} H_k(x) \\
&\quad + \sum_{r\geq 2} \sum_{k_1,\ldots,k_r\in Z^d} \frac{\beta^{[j]}_{k_1\cdots k_r}}{r} \{\{\cdots\{\{H_{k_1}, H_{k_2}\}, H_{k_3}\}\cdots\}, H_{k_r}\}(x), \\
\tilde{K}(x) &= H_0(x) + \sum_{r\geq 2} \sum_{k_1,\ldots,k_r\in Z^d} \frac{\bar{\beta}_{k_1\cdots k_r}}{r} \{\{\cdots\{\{H_{k_1}, H_{k_2}\}, H_{k_3}\}\cdots\}, H_{k_r}\}(x),
\end{aligned}$$

where the coefficients $\beta^{[j]}_{k_1\cdots k_r}$ and $\bar{\beta}_{k_1\cdots k_r}$ only depend on $\omega \in \mathbb{R}^d$.

Explicit recursive formulae for the coefficients

Given $r \in \mathbb{Z}^+$, $k \in \mathbb{Z}^d \backslash \{0\}$, and $l_1, \ldots, l_s \in \mathbb{Z}^d$,

$$
\begin{aligned}
\beta^{[j]}_{k} &= \frac{k_j}{k\cdot\omega}, \\
\beta^{[j]}_{0^r} &= 0, \\
\beta^{[j]}_{0^r k} &= \frac{i}{k\cdot\omega}\, \beta^{[j]}_{0^{r-1}k}, \\
\beta^{[j]}_{k l_1 \cdots l_s} &= \frac{i}{k\cdot\omega}(\beta^{[j]}_{l_1\cdots l_s} - \beta^{[j]}_{(k+l_1) l_2 \cdots l_s}), \\
\beta^{[j]}_{0^r k l_1 \cdots l_s} &= \frac{i}{k\cdot\omega}(\beta^{[j]}_{0^{r-1} k l_1\cdots l_s} - \beta^{[j]}_{0^r (k+l_1) l_2 \cdots l_s}),
\end{aligned}
$$

and very similar recursions for $\bar{\beta}_{k_1\cdots k_r}$. Also,

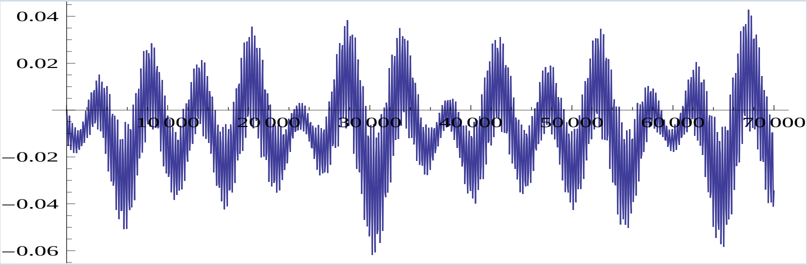
$$\bar{\beta}_{k_1\cdots k_r} = 1 - \sum_{j=1}^{d} \beta^{[j]}_{k_1\cdots k_r}.$$

Example (continuation)

We compute the second order truncation of $\tilde{I}_1(x)$,

$$\tilde{I}_1(x) = I_1(x) + \sum_{k \in \mathbb{Z}^d} \beta_k^{[1]} H_k + \sum_{k,\ell \in \mathbb{Z}^d} \beta_{k\ell}^{[1]} \{H_\ell, H_k\},$$

and plot $|\tilde{I}_1(x(t)) - \tilde{I}_1(x(0))|/\epsilon^5$ versus time.



Variation of similar size for $|\tilde{I}_1(x(t)) - \tilde{I}_1(x(0))|/\epsilon^5$ with $\epsilon = 1/140$.

A general class of highly oscillatory systems (HOS)

$$\frac{d}{dt}x = \sum_{j=1}^{d} \omega_j g_j(x) + h(x) \tag{1}$$

- non-resonant $\omega = (\omega_1, \ldots, \omega_d)$, and
- the t-flows $\Psi_t^{[j]}$ of the individual systems $\frac{d}{dt}x = g_j(x)$ are (2π)-periodic and commute with each other.

We denote for each $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{T}^d$,

$$\Psi_\theta = \Psi_{\theta_1}^{[1]} \circ \cdots \circ \Psi_{\theta_d}^{[d]}.$$

Obviously, $\forall (\theta, \theta') \in \mathbb{T}^d \times \mathbb{T}^d$,

$$\Psi_\theta \circ \Psi_{\theta'} = \Psi_{\theta+\theta'}.$$

$\{\Psi_\theta \ : \ \theta \in \mathbb{T}^d\}$ forms a d-parameter commutative group.

The change of variables $x = \Psi_{t\omega}(y)$ transforms the system into

$$\frac{d}{dt}y = f(y, t\omega) := \left(\frac{\partial}{\partial y}\Psi_{t\omega}(y)\right)^{-1} h(\Psi_{t\omega}(y)), \quad y(0) = x(0).$$

We consider the Fourier expansion

$$\left(\frac{\partial}{\partial y}\Psi_{\theta}(y)\right)^{-1} h(\Psi_{\theta}(y)) = \sum_{k\in\mathbb{Z}^d} e^{i(k\cdot\theta)} f_k(y).$$

We have that $y(t) = B(\gamma(t, t\omega), y(0))$, and since $\gamma(t, t\omega) \in \widehat{\mathcal{G}}$,

$$y(t) \quad = \quad y(0) + \sum_{r\geq 1} \sum_{k_1,\ldots,k_r\in Z^d} \gamma_{k_1\cdots k_r}(t, t\omega)\, f_{k_1\cdots k_r}(y(0)),$$

- where $f_{k_1\cdots k_r}(y) = \frac{\partial}{\partial y} f_{k_1\cdots k_r}(y) \cdot f_{k_r}(y)$, and
- each $\gamma_{k_1\cdots k_r}(t, \theta)$ depends polynomially in t and it is a trigonometric polynomial in each component of θ.

Recursive formulae for $\gamma_w(t,\theta)$

Given $r \in \mathbb{Z}^+$, $k \in \mathbb{Z}^d \setminus \{0\}$, and $l_1, \ldots, l_s \in \mathbb{Z}^d$,

$$\begin{aligned}
\gamma_k(t,\theta) &= \frac{i}{k\cdot\omega}(1-e^{i(k\cdot\theta)}),\\
\gamma_{0^r}(t,\theta) &= \frac{t^r}{r!},\\
\gamma_{0^r k}(t,\theta) &= \frac{i}{k\cdot\omega}(\gamma_{0^{r-1}k}(t,\theta)-\gamma_{0^r}(t,\theta)e^{i(k\cdot\theta)}),\\
\gamma_{kl_1\cdots l_s}(t,\theta) &= \frac{i}{k\cdot\omega}(\gamma_{l_1\cdots l_s}(t,\theta)-\gamma_{(k+l_1)l_2\cdots l_s}(t,\theta)),\\
\gamma_{0^r kl_1\cdots l_s}(t,\theta) &= \frac{i}{k\cdot\omega}(\gamma_{0^{r-1}kl_1\cdots l_s}(t,\theta)-\gamma_{0^r(k+l_1)l_2\cdots l_s}(t,\theta)).
\end{aligned}$$

With the notation $\Phi_{t,\theta}(x) := \Psi_\theta(B(\gamma(t,\theta),x))$, that is

$$\Phi_{t,\theta}(x) = \Psi_\theta\Big(x + \sum_{r\geq 1}\sum_{k_1,\ldots,k_r\in Z^d} \gamma_{k_1\cdots k_r}(t,\theta)\, f_{k_1\cdots k_r}(x)\Big),$$

we have that $x(t) = \Phi_{t,t\omega}(x(0))$ for any solution $x(t)$ of the original (autonomous) system, and thus, $\forall t, t' \in \mathbb{R}$,

$$\Phi_{t',t'\omega} \circ \Phi_{t,t\omega} = \Phi_{t+t',(t+t')\omega}.$$

Theorem

$\forall (t,\theta),(t',\theta') \in \mathbb{R} \times \mathbb{T}^d$

$$\Phi_{t',\theta'} \circ \Phi_{t,\theta} = \Phi_{t+t',\theta+\theta'}.$$

Hence, $\{\Phi_{t,\theta} \ : \ (t,\theta) \in \mathbb{R} \times \mathbb{T}^d\}$ forms a $(d+1)$-parameter commutative group of formal transformations.

Consider $\bar{\Phi}_t(x) := \Phi_{t,0}(x) = B(\gamma(t,0), x)$. Since $\bar{\Phi}_t \circ \bar{\Phi}_{t'} = \bar{\Phi}_{t+t'}$, $\bar{\Phi}_t$ is the t-flow of an autonomous ODE $\frac{d}{dt}x = \tilde{h}(x)$, where

$$\tilde{h}(x) \quad = \quad \left.\frac{d}{dt}\Phi_{t,0}(x)\right|_{t=0}.$$

That is,

$$\tilde{h}(x) \quad = \quad B(\bar{\beta}, x), \quad \bar{\beta} = \left.\frac{d}{dt}\gamma(t,0)\right|_{t=0}.$$

Moreover, $\gamma(t,0) \in \widehat{\mathcal{G}}$, which implies that

$$\tilde{h}(x) \quad = \quad \sum_{r \geq 1} \sum_{k_1,\ldots,k_r \in Z^d} \bar{\beta}_{k_1\cdots k_r} \, f_{k_1\cdots k_r}(x),$$

Theorem

For any solution $x(t)$ of the HOS,

$$x(t) = \Phi_{0,t\omega}(X(t)),$$

where $X(t)$ is the solution of the averaged system

$$\frac{d}{dt}X = \tilde{h}(X), \quad X(0) = x(0).$$

Moreover, if $I(x)$ is a first integral of the original HOS,

$$I(x(t)) \equiv I(x(0)),$$

then $I(X(t)) \equiv I(X(0))$.

Some comments on that high order averaging procedure:

- The name averaging: A first order approximation of $\tilde{h}(X)$ is given simply by

$$f_0(X) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \left(\frac{\partial}{\partial X} \Psi_\theta(X) \right)^{-1} h(\Psi_\theta(X))\, d\theta.$$

- In the periodic case ($d = 1$, $\omega \in \mathbb{R}$), it is called stroboscopic averaging, because $x(t_n) = X(t_n)$ at $t_n = 2\pi n/\omega$.
- In the quasi-periodic case ($d > 1$, $\omega \in \mathbb{R}^d$), we refer to it as quasi-stroboscopic averaging, because $\forall \delta > 0$, there exist $T(\delta) > 0$ and $k(\delta) \in \mathbb{Z}^d$ such that $T(\delta)\,\omega = 2\pi k(\delta) + \mathcal{O}(\delta)$ and thus, $x(t_n) = X(t_n) + \mathcal{O}(n\delta)$ at quasi-stroboscopic times $t_n = nT(\delta)$.

The precedent theorem can be supplemented as follows:

Theorem

1. *The change of variables* $x = \Psi_{0,t\omega}(X)$ *admits the factorization*

$$\Phi_{0,\omega t} = \Phi^{[1]}_{t\omega_1} \circ \cdots \circ \Phi^{[d]}_{t\omega_d},$$

where each $\Psi^{[j]}_t$ *is the* $((2\pi)$*-periodic) t-flow of*

$$\frac{d}{dt}x = \tilde{g}_j(x), \quad \textit{where} \quad \tilde{g}_j(x) = g_j(x) + B(\beta^{[j]}, x),$$

with $\beta^{[j]} = \frac{d}{dt}\gamma(0, te_j)\big|_{t=0}$ *(*e_j *the* j*th unit vector in* $\mathbb{R}^d$*).*

2. *If* $I(x(t)) \equiv I(x(0))$*, then* $I(x)$ *is also a first integral of each vector field* $\tilde{g}_j(x)$*.*

The factorization

$$\Phi_{t,\omega t} = \Phi_{t,0} \circ \Phi^{[1]}_{t\omega_1} \circ \cdots \circ \Phi^{[d]}_{t\omega_d}$$

of the t-flow of the HOS in commuting flows gives the following

Rewriting of the HOS

$$\frac{d}{dt}x \quad = \quad \sum_{j=1}^{d} \omega_j g_j(x) + r(x) \; \equiv \; \sum_{j=1}^{d} \omega_j \tilde{g}_j(x) + \tilde{h}(x)$$

where $\tilde{g}_j(x)$ $(1 \leq j \leq d)$ and $\tilde{h}(x)$ commute with each other. (Closely related to normal forms).