Discrete Dirac Mechanics and Discrete Dirac Geometry

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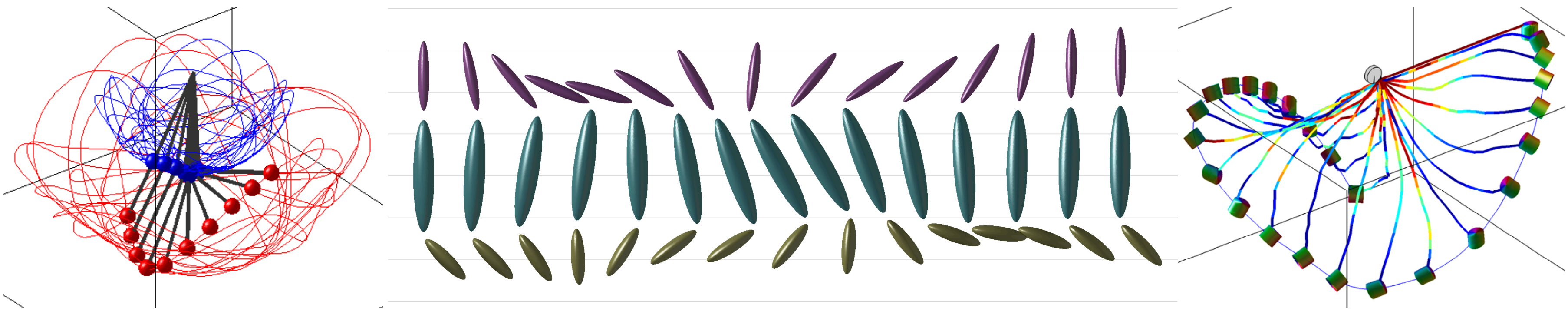
Motivation

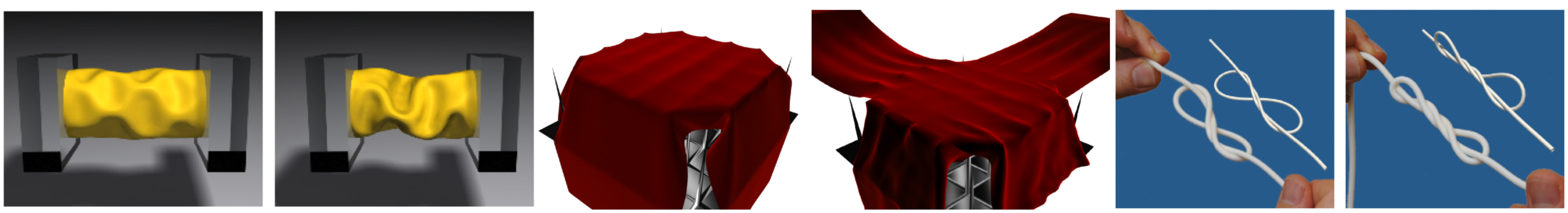
■ Variational Integrators for Multi-Physics Simulations

- Multibody Systems



Simulations courtesy of Taeyoung Lee, Florida Institute of Technology.

- Continuum Mechanics

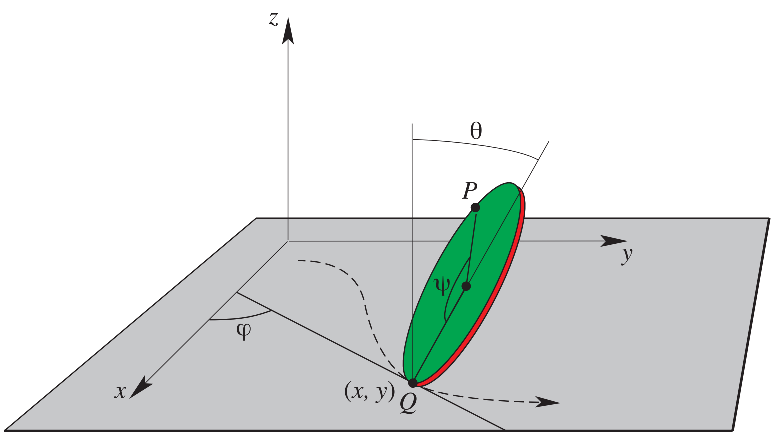


Simulations courtesy of Eitan Grinspun, Columbia.

Introduction

Dirac Structures

- Dirac structures can be viewed as simultaneous generalizations of symplectic and Poisson structures.
- Implicit Lagrangian and Hamiltonian systems[1] provide a unified geometric framework for studying degenerate, interconnected, and nonholonomic Lagrangian and Hamiltonian mechanics.
- The category of Lagrange–Dirac systems are closed under interconnection, which allows for the distributed parallel implementation of hierarchical multiphysics models using the interconnection paradigm.



[1] H. Yoshimura, J.E. Marsden, Dirac structures in Lagrangian mechanics. Part I: Implicit Lagrangian systems, *J. of Geometry and Physics*, **57**, 133–156, 2006.

Introduction

Variational Principles

- The Hamilton–Pontryagin principle[2] on the Pontryagin bundle $TQ \oplus T^*Q$, unifies Hamilton's principle, Hamilton's phase space principle, and the Lagrange–d'Alembert principle.
- Provides a variational characterization of implicit Lagrangian and Hamiltonian systems.
- This naturally leads to the development of variational integration techniques for interconnected systems.

[2] H. Yoshimura, J.E. Marsden, Dirac structures in Lagrangian mechanics. Part II: Variational structures, *J. of Geometry and Physics*, **57**, 209–250, 2006.

Dirac Structures on Vector Spaces

Properties

- Let V be a n-dimensional vector space, with the pairing $\langle\langle \cdot\, , \, \cdot \rangle\rangle$ on $V \oplus V^*$ given by

$$\langle\langle (v, \alpha), (\tilde{v}, \tilde{\alpha}) \rangle\rangle = \langle \alpha, \tilde{v} \rangle + \langle \tilde{\alpha}, v \rangle,$$

where $\langle \cdot\, , \, \cdot \rangle$ is the natural pairing between covectors and vectors.

- A **Dirac Structure** is a subspace $D \subset V \oplus V^*$, such that

$$D = D^{\perp}.$$

- In particular, $D \subset V \oplus V^*$ is a Dirac structure iff

$$\dim D = n, \quad \text{and} \quad \langle \alpha, \tilde{v} \rangle + \langle \tilde{\alpha}, v \rangle = 0,$$

for all $(v, \alpha), (\tilde{v}, \tilde{\alpha}) \in D$.

Dirac Structures on Manifolds

Generalizing Symplectic and Poisson Structures

- Let $M = T^*Q$.
- The graph of the symplectic two-form $\Omega : TM \times TM \to \mathbb{R}$, viewed as a map $TM \to T^*M$,

$$v_z \mapsto \Omega(v_z, \cdot),$$

  is a Dirac structure.
- Similarly, the graph of the Poisson structure $B : T^*M \times T^*M \to \mathbb{R}$, viewed as a map T^*M to $T^{**}M \cong TM$,

$$\alpha_z \mapsto B(\alpha_z, \cdot),$$

  is a Dirac structure.
- Furthermore, if the symplectic form and the Poisson structure are related, they induce the same Dirac structure on $TM \oplus T^*M$.

Motivating Example: Electrical Circuits

Configuration space and constraints

- The **configuration** $q \in E$ of the electrical circuit is given by specifying the current in each branch of the electrical circuit.
- Not all configurations are admissible, due to **Kirchhoff's Current Laws**:

  the sum of currents at a junction is zero.

  This induce a **constraint KCL space** $\Delta \subset TE$.
- Its annihilator space $\Delta^\circ \subset T^*E$ is defined by

$$\Delta_q^\circ = \{e \in T_q^*E \,|\, \langle e, f\rangle = 0 \text{ for all } f \in \Delta_q\},$$

  which can be identified with the set of **branch voltages**, and encodes the **Kirchhoff's Voltage Laws**:

  the sum of voltages about a closed loop is zero.

Motivating Example: Electrical Circuits

Dirac structures and Tellegen's theorem

- Given $\Delta \subset TE$ and $\Delta^\circ \subset T^*E$ which encode the Kirchhoff's current and voltage laws,

$$D_E = \Delta \oplus \Delta^\circ \subset TE \oplus T^*E$$

  is a Dirac structure on E.

- Since $D = D^\perp$, we have that for each $(f, e) \in D_E$,

$$\langle e, f \rangle = 0.$$

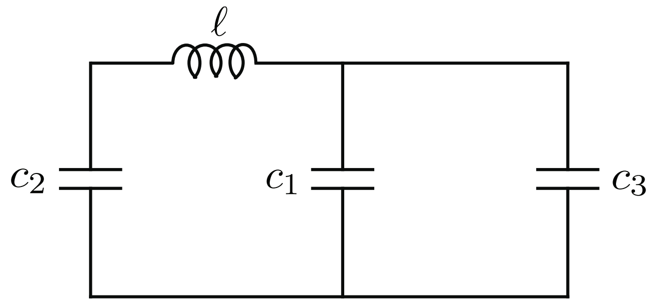
  This is a statement of **Tellegen's theorem**, which is an important result in the network theory of circuits.

Motivating Example: Electrical Circuits

Lagrangian for LC-circuits

- **Dirac's theory of constraints** was concerned with degenerate Lagrangians where the set of **primary constraints**, the image $P \subset T^*Q$ of the Legendre transformation, is not the whole space.
- **Magnetic energy**

$$T(f) = \sum \frac{1}{2} L_i f_{L_i}^2.$$



- **Electric potential energy**

$$V(q) = \sum \frac{1}{2} \frac{q_{C_i}^2}{C_i}.$$

- **Lagrangian**

$$L(q, f) = T(f) - V(q).$$

$$L(q, f) = \frac{\ell}{2}(f^\ell)^2 - \frac{(q^{c_1})^2}{2c_1} - \frac{(q^{c_2})^2}{2c_2} - \frac{(q^{c_3})^2}{2c_3},$$

$$\Delta_Q = \{f \in TQ \mid \omega^a(f) = 0,\ a = 1, 2\},$$

$$\omega^1 = -dq^\ell + dq^{c_2},$$

$$\omega^2 = -dq^{c_1} + dq^{c_2} - dq^{c_3}.$$

Interconnected Systems

Interconnection of Lagrange–Dirac Systems

- Given two Lagrange–Dirac systems: (Q_1, L_1, Δ_1) and (Q_2, L_2, Δ_2), the implicit Euler–Lagrange equations are given by,

$$(X_1, \mathbf{d}E_1|_{P_1}) \in D_{\Delta_1}, \qquad (X_2, \mathbf{d}E_2|_{P_2}) \in D_{\Delta_2},$$

where the **generalized energy** E_i is defined as

$$E_i(q_i, \dot{q}_i, p_i) = \langle p_i, \dot{q}_i \rangle - L(q_i, \dot{q}_i).$$

- The **interconnection** of these two systems is a Lagrange–Dirac system on the product $Q = Q_1 \times Q_2$ with Lagrangian,

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = L_1(q_1, \dot{q}_1) + L_2(q_2, \dot{q}_2),$$

with constraints

$$\Delta = (\Delta_1 \times \Delta_2) \cap \Delta_{int}.$$

Interconnected Systems

Interconnection Dirac Structures

- Interconnected Lagrange–Dirac systems can be understood using the Lagrange–d'Alembert–Pontryagin principle.
- It is also interesting to express it in terms of an **interconnection Dirac structure** D_c,

$$(X_1 \times X_2, \mathbf{d}(E_1 + E_2)|_P) \in D_c,$$

where $D_c = (D_{\Delta_1} \oplus D_{\Delta_2}) \bowtie D_{int}$, and the **bowtie construction** $\bowtie$ is given by

$$D_A \bowtie D_B := \{(w, \alpha) \in TT^*Q \oplus T^*T^*Q \mid w \in \tau_{TT^*Q}(D_A \cap D_B), \\ \alpha - \Omega^\flat_{TT^*Q} w \in \left(\tau_{TT^*Q}(D_A \cap D_B)\right)^\circ\}.$$

Variational Principles

Continuous Hamilton–Pontryagin principle

Pontryagin bundle and Hamilton–Pontryagin principle

- Consider the **Pontryagin bundle** $TQ \oplus T^*Q$, which has local coordinates (q, v, p).
- The **Hamilton–Pontryagin principle** is given by

$$\delta \int [L(q, v) - p(v - \dot{q})] = 0,$$

where we impose the second-order curve condition, $v = \dot{q}$ using Lagrange multipliers p.

Continuous Hamilton–Pontryagin principle

Implicit Lagrangian systems

- Taking variations in q, v, and p yield

$$\delta \int [L(q,v) - p(v - \dot{q})] dt$$
$$= \int \left[\frac{\partial L}{\partial q} \delta q + \left(\frac{\partial L}{\partial v} - p \right) \delta v - (v - \dot{q}) \delta p + p \delta \dot{q} \right] dt$$
$$= \int \left[\left(\frac{\partial L}{\partial q} - \dot{p} \right) \delta q + \left(\frac{\partial L}{\partial v} - p \right) \delta v - (v - \dot{q}) \delta p \right] dt$$

where we used integration by parts, and the fact that the variation δq vanishes at the endpoints.

- This recovers the **implicit Euler–Lagrange equations**,

$$\dot{p} = \frac{\partial L}{\partial q}, \qquad p = \frac{\partial L}{\partial v}, \qquad v = \dot{q}.$$

Continuous Hamilton–Pontryagin principle

Hamilton's phase space principle

- By taking variations with respect to v, we obtain the **Legendre transform**,

$$\frac{\partial L}{\partial v}(q,v) - p = 0.$$

- The **Hamiltonian**, $H : T^*Q \to \mathbb{R}$, is defined to be,

$$H(q,p) = \operatorname*{ext}_{v} \Big(pv - L(q,v) \Big) = pv - L(q,v)\big|_{p=\partial L/\partial v(q,v)} .$$

- The Hamilton–Pontryagin principle reduces to,

$$\delta \int [p\dot{q} - H(q,p)] = 0,$$

which is the **Hamilton's principle in phase space**.

Continuous Hamilton–Pontryagin principle

Lagrange–d'Alembert–Pontryagin principle

- Consider a constraint distribution $\Delta_Q \subset TQ$.
- The **Lagrange–d'Alembert–Pontryagin principle** is given by

$$\delta \int L(q, v) - p(v - \dot{q})dt = 0,$$

for fixed endpoints, and variations $(\delta q, \delta v, \delta p)$ of $(q, v, p) \in TQ \oplus T^*Q$, such that $(\delta q, \delta v) \in (T\tau_Q)^{-1}(\Delta_Q)$, where $\tau_Q : TQ \to Q$.
- This gives the **implicit Euler–Lagrange equations**,

$$\dot{q} = v \in \Delta_Q(q), \qquad p = \frac{\partial L}{\partial v}, \qquad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^\circ(q).$$

- This describes degenerate nonholonomic Lagrangian mechanics.

Discrete Variational Principles

Discrete Hamilton–Pontryagin principle

Discrete Pontryagin bundle and Hamilton–Pontryagin principle

- Consider the **discrete Pontryagin bundle** $(Q \times Q) \oplus T^*Q$, which has local coordinates (q_k^0, q_k^1, p_k).
- The **discrete Hamilton–Pontryagin principle** is given by

$$\delta \sum \left[L_d(q_k^0, q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0) \right] = 0,$$

  where we impose the second-order curve condition, $q_k^1 = q_{k+1}^0$ using Lagrange multipliers p_{k+1}
- The **discrete Lagrangian** L_d is a Type I generating function, and is chosen to be an approximation of **Jacobi's solution** of the Hamilton–Jacobi equation.

Discrete Hamilton–Pontryagin principle

Implicit discrete Lagrangian systems

- Taking variations in q_k^0, q_k^1, and p_k yield

$$\delta \sum \left[L_d(q_k^0, q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0) \right] = \sum \Big\{ [D_1 L_d(q_k^0, q_k^1) + p_k] \delta q_k^0 - [q_k^1 - q_{k+1}^0] \delta p_{k+1} + [D_2 L_d(q_k^0, q_k^1) - p_{k+1}] \delta q_k^1 \Big\} .$$

- This recovers the **implicit discrete Euler–Lagrange equations**,

$$p_k = -D_1 L_d(q_k^0, q_k^1), \qquad p_{k+1} = D_2 L_d(q_k^0, q_k^1), \qquad q_k^1 = q_{k+1}^0 .$$

Discrete Hamilton–Pontryagin principle

Discrete Hamilton's phase space principle

- By taking variations with respect to q_k^1, we obtain the **discrete Legendre transform**,

$$D_2 L_d(q_k^0, q_k^1) - p_{k+1} = 0$$

- The **discrete Hamiltonian**, $H_{d+} : Q \times Q^* \to \mathbb{R}$, is given by,

$$\begin{aligned} H_{d+}(q_k^0, p_{k+1}) &= \operatorname*{ext}_{q_k^1} \; p_{k+1} q_k^1 - L_d(q_k^0, q_k^1) \\ &= p_{k+1} q_k^1 - L_d(q_k^0, q_k^1) \Big|_{p_{k+1} = D_2 L_d(q_k^0, q_k^1)} . \end{aligned}$$

- The discrete Hamilton–Pontryagin principle reduces to,

$$\delta \sum [p_{k+1} q_{k+1} - H_{d+}(q_k, p_{k+1})] = 0,$$

which is the **discrete Hamilton's principle in phase space**.

Discrete Hamilton's Equations and Discrete Hamiltonians

Discrete Hamilton's Equations

- The discrete Hamilton's principle in phase space yields the following **discrete Hamilton's equations**,

$$p_k = D_1 H_{d+}(q_k, p_{k+1}), \qquad q_{k+1} = D_2 H_{d+}(q_k, p_{k+1})$$

- From this, it is clear that the discrete Hamiltonian H_{d+} is a Type II generating function of a symplectic transformation.
- The discrete Hamiltonian should approximate the **exact discrete Hamiltonian**, given by,

$$H^+_{d,\text{exact}}(q_k, p_{k+1}) = \underset{\substack{(q,p)\in C^2([t_k,t_{k+1}],T^*Q) \\ q(t_k)=q_k, p(t_{k+1})=p_{k+1}}}{\text{ext}} p(t_{k+1})q(t_{k+1}) - \int_{t_k}^{t_{k+1}} [p\dot{q} - H(q,p)]\, dt.$$

Nonintegrable Constraints

Discrete Constraint Distributions

- Given a continuous constraint distribution is $\Delta_Q \subset TQ$, let $\Delta_Q^\circ \subset T*Q$ be the annihilator codistribution, with a basis $\{\omega^a\}_{a=1}^m$.
- Given a **retraction** $\mathcal{R} : TQ \to Q$, we define functions $\omega_{d+}^a : Q \times Q \to \mathbb{R}$,

$$\omega_{d+}^a(q_0, q_1) := \omega^a \left(q_0, \mathcal{R}_{q_0}^{-1}(q_1) \right) .$$

- A **discrete constraint distribution** $\Delta_Q^{d+} \subset Q \times Q$ is,

$$\Delta_Q^{d+} := \left\{ (q_0, q_1) \in Q \times Q \mid \omega_{d+}^a(q_0, q_1) = 0,\ a = 1, 2, \ldots, m \right\} .$$

Nonintegrable Constraints

Discrete Lagrange–d'Alembert–Pontryagin Principle

- The **discrete Lagrange–d'Alembert–Pontryagin principle** is

$$\delta \sum_{k=0}^{N-1} \left[L_{\mathrm{d}}(q_k, q_k^+) + p_{k+1}(q_{k+1} - q_k^+) \right] = 0,$$

where we require $(q_k, q_{k+1}) \in \Delta_Q^{\mathrm{d}+}$; furthermore, the variations $(\delta q_k, \delta q_k^+, \delta p_{k+1})$ of (q_k, q_k^+, p_{k+1}) in $(Q \times Q) \oplus (Q \times Q^*)$ are assumed to satisfy $\delta q_k \in \Delta_Q(q_k)$, and $\delta q_0 = \delta q_N = 0$ at the endpoints.

Extension to Manifolds

Retraction

- A **retraction** on a manifold Q is a smooth mapping $\mathcal{R} : TQ \to Q$ with the following properties:
  - Let $\mathcal{R}_q : T_qQ \to Q$ be the restriction of $\mathcal{R}$ to T_qQ.
  - $\mathcal{R}_q(0_q) = q$, where 0_q denotes the zero element of T_qQ.
  - With the identification $T_{0_q}T_qQ \simeq T_qQ$, $\mathcal{R}_q$ satisfies

$$T_{0_q}\mathcal{R}_q = \mathrm{id}_{T_qQ},$$

where $T_{0_q}\mathcal{R}_q$ is the tangent map of $\mathcal{R}_q$ at $0_q \in T_qQ$.

Extension to Manifolds

Retraction Compatible Charts

- Let Q be an n-dimensional manifold equipped with a retraction $\mathcal{R} : TQ \to Q$. A coordinate chart (U, φ), $U \subset Q$ and $\varphi : U \to \mathbb{R}^n$ is said to be **retraction compatible at** $q \in U$ if
  - φ is centered at q, i.e., $\varphi(q) = 0$;
  - The compatibility condition,
$$\mathcal{R}(v_q) = \varphi^{-1} \circ T_q\varphi(v_q),$$
holds, where we identified $T_0\mathbb{R}^n$ with $\mathbb{R}^n$ as follows: Let $\varphi = (x^1, \ldots, x^n)$ with $x^i : U \to \mathbb{R}$ for $i = 1, \ldots, n$. Then
$$v^i \frac{\partial}{\partial x^i} \mapsto (v^1, \ldots, v^n).$$
- A coordinate map φ can be obtained using the identification $\mathcal{R}_q : T_qQ \to Q$, and coordinatizing T_qQ by introducing a basis.

Extension to Manifolds

Retraction Compatible Atlas

- An atlas for the manifold Q is **retraction compatible** if it consists of retraction compatible coordinate charts.

Lie Group Example

- On a Lie group G, consider the exponential map $\exp : \mathfrak{g} \to G$.
- Then, the map $\mathcal{R}_g : T_g \to G$,

$$\mathcal{R}_g := L_g \circ \exp \circ T_g L_{g^{-1}},$$

  is a retraction.
- Furthermore, canonical coordinates of the first kind, $\psi_g : U_g \to \mathfrak{g}$

$$\psi_g := \exp^{-1} \circ L_{g^{-1}},$$

  are retraction compatible.

Extension to Manifolds

Discrete Lagrange–d'Alembert–Pontryagin Principle with Retraction

- The **discrete Lagrange–d'Alembert–Pontryagin principle with Retraction** is

$$\delta \sum_{k=0}^{N-1} \left\{ L_d(q_k, q_k^+) + \left\langle p_{k+1}, \mathcal{R}^{-1}_{q_{k+1}}(q_{k+1}) - \mathcal{R}^{-1}_{q_{k+1}}(q_k^+) \right\rangle \right\} = 0,$$

  where we require $(q_k, q_{k+1}) \in \Delta_Q^{d+}$; furthermore, the variations $(\delta q_k, \delta q_k^+, \delta p_{k+1})$ of (q_k, q_k^+, p_{k+1}) in $Q \times T^*Q$ are assumed to satisfy $\delta q_k \in \Delta_Q(q_k)$, and also $\delta q_0 = \delta q_N = 0$ at the endpoints.

- This variational principle is well-defined semi-globally, and on a retraction-compatible chart, it reduces to the local expression for the discrete Lagrange–d'Alembert–Pontryagin principle.

Extension to Manifolds

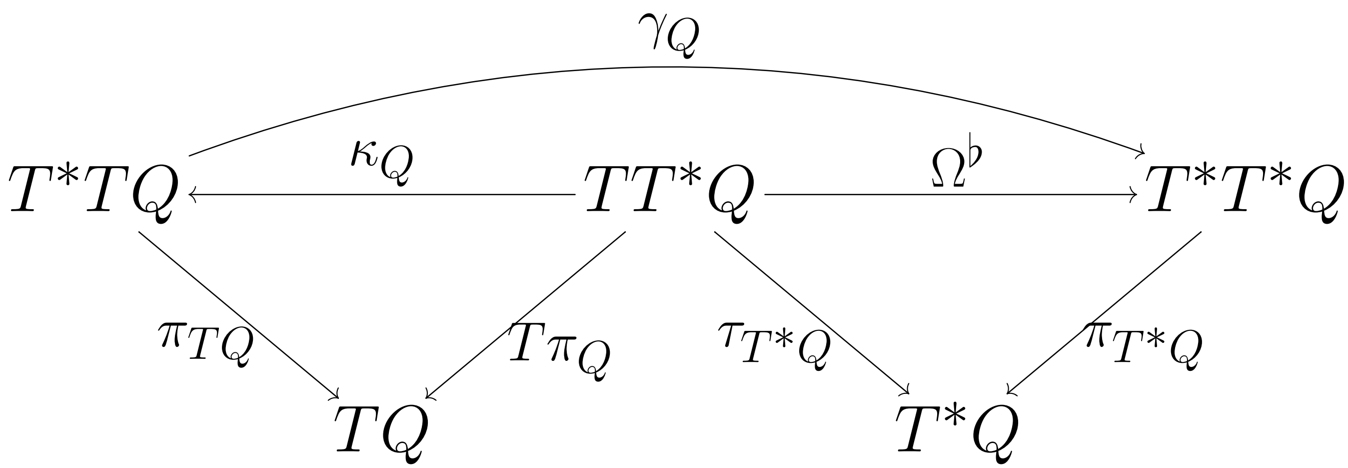
Implications for discrete Dirac structures

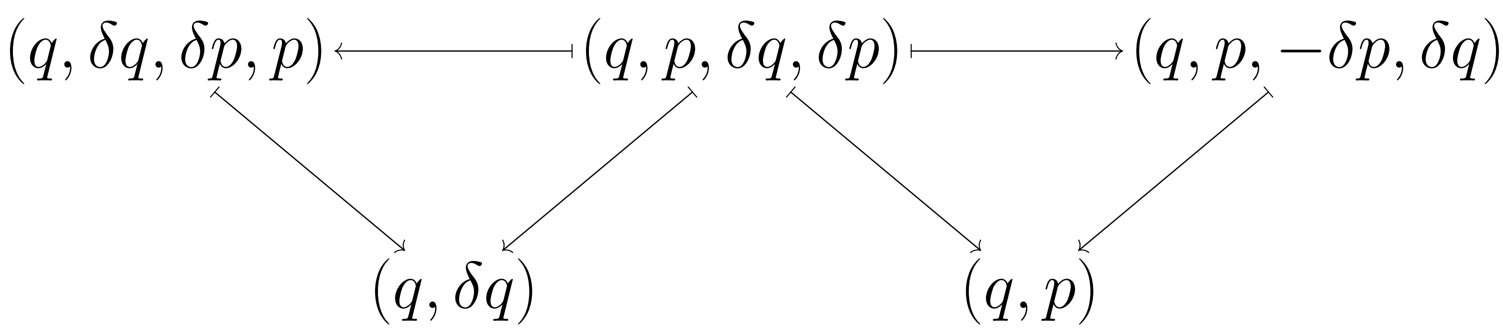
- The map $(\pi_Q, \mathcal{R}) : TQ \to Q \times Q$ induces a local diffeomorphism of the continuous Pontryagin bundle $TQ \oplus T^*Q$ with the discrete Pontryagin bundle $(Q \times Q) \times_Q T^*Q$.
- This identification then induces a discrete Dirac structure.
- We have also developed an equivalent characterization of discrete Dirac structures that more clearly elucidates the role of the geometry of symplectic maps in induced discrete Dirac structures.

Dirac Structures

Continuous Dirac Mechanics

Tulczyjew Triple





Continuous Dirac Mechanics

Dirac Structures and Constraints

- A constraint distribution $\Delta_Q \subset TQ$ induces a **Dirac structure** on T^*Q,

$$D_{\Delta_Q}(z) := \Big\{ (v_z, \alpha_z) \in T_z T^*Q \times T_z^* T^*Q \;\Big|\; v_z \in \Delta_{T^*Q}(z),\ \alpha_z - \Omega^\flat(v_z) \in \Delta^\circ_{T^*Q}(z) \Big\}$$

where $\Delta_{T^*Q} := (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q$.

- Holonomic and nonholonomic constraints, as well as constraints arising from interconnections can be incorporated into the Dirac structure.

Continuous Dirac Mechanics

Implicit Lagrangian Systems

- Let $\gamma_Q := \Omega^\flat \circ (\kappa_Q)^{-1} : T^*TQ \to T^*T^*Q$.
- Given a Lagrangian $L : TQ \to \mathbb{R}$, define $\mathfrak{D}L := \gamma_Q \circ dL$.
- An **implicit Lagrangian system** (L, Δ_Q, X) is,

$$(X, \mathfrak{D}L) \in D_{\Delta_Q},$$

where $X \in \mathfrak{X}(T^*Q)$.

- This gives the **implicit Euler–Lagrange equations**,

$$\dot{q} = v \in \Delta_Q(q), \qquad p = \frac{\partial L}{\partial v}, \qquad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^\circ(q).$$

- In the special case $\Delta_Q = TQ$, we obtain,

$$\dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}, \quad p = \frac{\partial L}{\partial v}.$$

Continuous Dirac Mechanics

Implicit Hamiltonian Systems

- Given a Hamiltonian $H : T^*Q \to \mathbb{R}$, an **implicit Hamiltonian system** (H, Δ_Q, X) is,

$$(X, dH) \in D_{\Delta_Q},$$

which gives the **implicit Hamilton's equations**,

$$\dot{q} = \frac{\partial H}{\partial p} \in \Delta_Q(q), \qquad \dot{p} + \frac{\partial H}{\partial q} \in \Delta_Q^\circ(q).$$

- In the special case $\Delta_Q = TQ$, we recover the standard Hamilton's equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

Discrete Dirac Structures

The Geometry of Generating Functions

Generating Functions of Type I and the κ_Q^d map

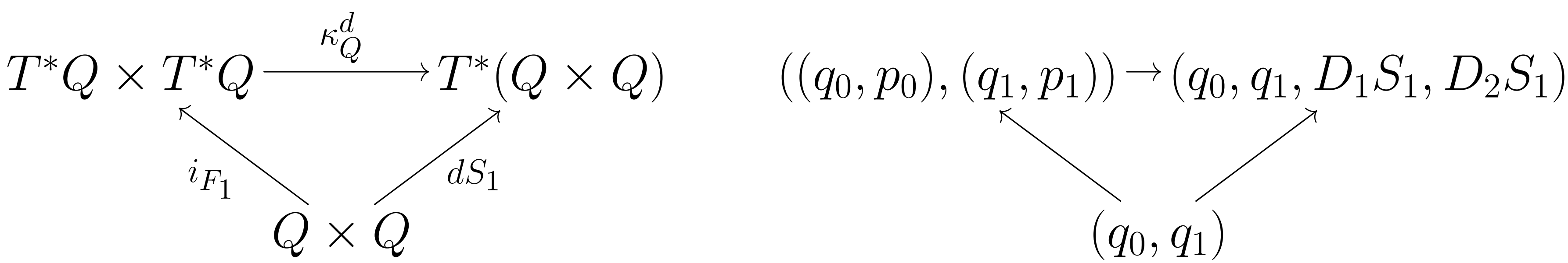
- The flow F_1 on T^*Q is symplectic iff there exists $S_1 : Q \times Q \to \mathbb{R}$,

$$(i_{F_1})^* \Theta_{T^*Q \times T^*Q} = dS_1.$$

which gives

$$p_0 = -D_1 S_1, \qquad p_1 = D_2 S_1.$$

- We require that the following diagram commutes,



- This gives rise to a map $\kappa_Q^d : T^*Q \times T^*Q \to T^*(Q \times Q)$

$$\kappa_Q^d : ((q_0, p_0), (q_1, p_1)) \mapsto (q_0, q_1, -p_0, p_1).$$

The Geometry of Generating Functions

Generating Functions of Type II and the $\Omega^\flat_{d+}$ map

- The flow F_2 on T^*Q is symplectic iff there exists $S_2 : Q \times Q^* \to \mathbb{R}$,

$$(i_{F_2})^* \Theta^{(2)}_{T^*Q \times T^*Q} = dS_2,$$

which gives

$$p_0 = D_1 S_2, \qquad q_1 = D_2 S_2.$$

- We require that the following diagram commutes,

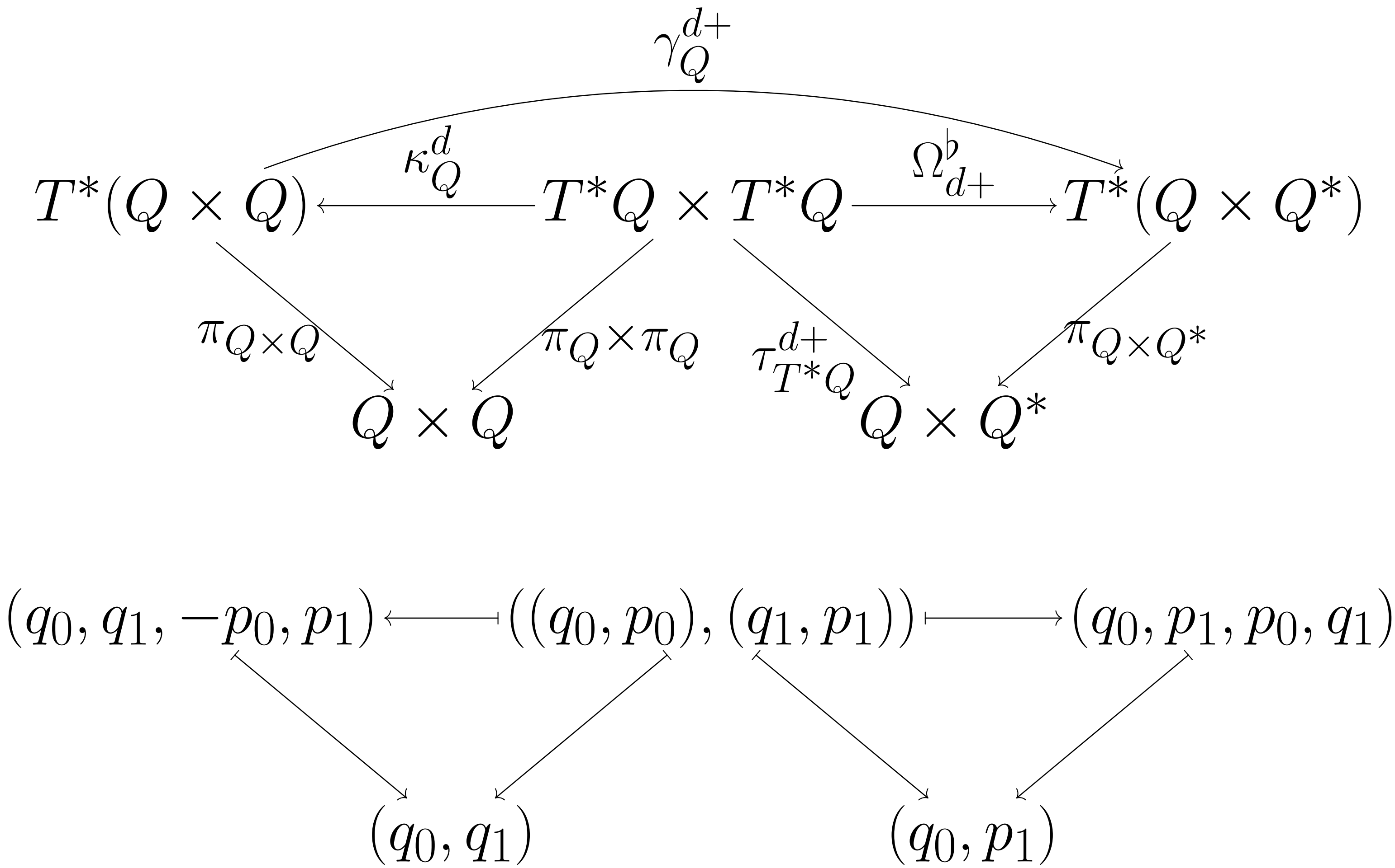
$$\begin{array}{ccccc} T^*Q \times T^*Q & & \xrightarrow{\;\Omega^\flat_{d+}\;} & & T^*(Q \times Q^*) \\ & \nwarrow^{i_{F_2}} & & \nearrow^{dS_2} & \\ & & Q \times Q^* & & \end{array} \qquad \begin{array}{ccccc} ((q_0, p_0), (q_1, p_1)) & & \to & & (q_0, p_1, D_1 S_2, D_2 S_2) \\ & \nwarrow & & \nearrow & \\ & & (q_0, p_1) & & \end{array}$$

- This gives rise to a map $\Omega^\flat_{d+} : T^*Q \times T^*Q \to T^*(Q \times Q^*)$

$$\Omega^\flat_{d+} : ((q_0, p_0), (q_1, p_1)) \mapsto (q_0, p_1, p_0, q_1).$$

(+)-Discrete Dirac Mechanics

■ Discrete Tulczyjew Triple



(+)-Discrete Dirac Mechanics

Discrete Induced Dirac Structure

- Define the **discrete induced Dirac structure** $D^{d+}_{\Delta_Q} \subset (T^*Q \times T^*Q) \times T^*(Q \times Q^*)$ by

$$D^{d+}_{\Delta_Q} := \Big\{ ((z, z^+), \alpha_{\hat{z}}) \in (T^*Q \times T^*Q) \times T^*(Q \times Q^*) \mid \left(z, z^+\right) \in \Delta^{d+}_{T^*Q},\ \alpha_{\hat{z}} - \Omega^\flat_{d+}\left((z, z^+)\right) \in \Delta^\circ_{Q \times Q^*} \Big\},$$

where,

$$\Delta^{d+}_{T^*Q} := \Big\{ ((q_0, p_0), (q_1, p_1)) \in T^*Q \times T^*Q \mid (q_0, q_1) \in \Delta^{d+}_Q \Big\},$$

$$\Delta^\circ_{Q \times Q^*} := \Big\{ (q, p, \alpha_q, 0) \in T^*(Q \times Q^*) \mid \alpha_q \in \Delta^\circ_Q(q) \Big\}.$$

(+)-Discrete Dirac Mechanics

Implicit Discrete Lagrangian Systems

- Let $\gamma_Q^{d+} := \Omega_{d+}^{\flat} \circ (\kappa_Q^d)^{-1} : T^*(Q \times Q) \to T^*(Q \times Q^*)$.
- Given a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$, define $\mathfrak{D}^+ L_d := \gamma_Q^{d+} \circ dL$.
- An **implicit discrete Lagrangian system** is given by

$$\left(X_d^k, \mathfrak{D}^+ L_d(q_k^0, q_k^1)\right) \in D_{\Delta_Q}^{d+},$$

where $X_d^k = ((q_k^0, p_k^0), (q_{k+1}^0, p_{k+1}^0)) \in T^*Q \times T^*Q$.

- This gives the **implicit discrete Euler–Lagrange equations**,

$$p_{k+1}^0 = D_2 L_d(q_k^0, q_k^1) \in \Delta_Q^\circ(q_k^1), \qquad p_k^0 + D_1 L_d(q_k^0, q_k^1) \in \Delta_Q^\circ(q_k^0),$$

$$q_k^1 = q_{k+1}^0, \qquad (q_k^0, q_{k+1}^0) \in \Delta_Q^d.$$

(+)-Discrete Dirac Mechanics

Implicit Discrete Hamiltonian Systems

- Given a discrete Hamiltonian $H_{d+} : Q \times Q^* \to \mathbb{R}$, an **implicit discrete Hamiltonian system** $(H_{d+}, \Delta_Q^d, X_d)$ is,

$$\left(X_d^k, dH_{d+}(q_k^0, p_k^1)\right) \in D_{\Delta_Q}^{d+},$$

which gives the **implicit discrete Hamilton's equations**,

$$p_k^0 - D_1 H_{d+}(q_k^0, p_k^1) \in \Delta_Q^\circ(q_k^0), \qquad q_{k+1}^0 = D_2 H_{d+}(q_k^0, p_k^1),$$

$$p_k^1 - p_{k+1}^0 \in \Delta_Q^\circ(q_k^1), \qquad (q_k^0, q_{k+1}^0) \in \Delta_Q^d,$$

Conclusion

Discrete Dirac Structures

- We have constructed a discrete analogue of a Dirac structure by considering the geometry of generating functions of symplectic maps.
- Unifies geometric integrators for degenerate, interconnected, and nonholonomic Lagrangian and Hamiltonian systems.
- Provides a characterization of the discrete geometric structure associated with Hamilton–Pontryagin integrators.

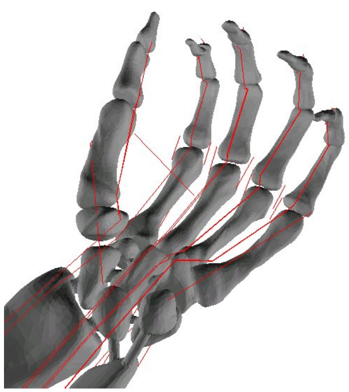
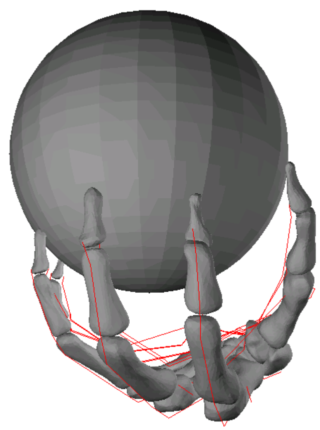
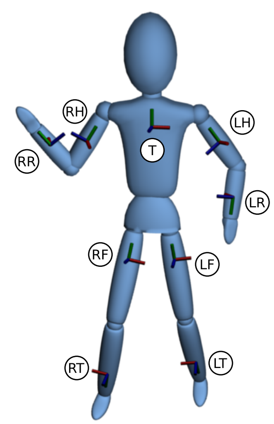
Discrete Hamilton–Pontryagin principle

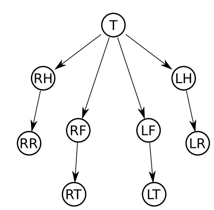
- Provides a unified discrete variational principle that recovers both the discrete Hamilton's principle, and the discrete Hamilton's phase space principle.
- Is sufficiently general to characterize all near to identity Dirac maps.

Conclusion

Current Work and Future Directions

- Extend the discrete Dirac approach to interconnected systems, and develop modular and parallel implementations.
- Develop generalizations to Hamiltonian PDEs: discrete analogues of multi-Dirac structures, and multi-Dirac mechanics.
- Derive the Dirac analogue of the Hamilton–Jacobi equation, with nonholonomic Hamilton–Jacobi theory as a special case.



Simulations courtesy of Todd Murphey, Northwestern University.