Asymptotic numerical solvers for oscillatory systems of differential equations

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Outline

- Motivation
- Filon-type methods
- Asymptotic numerical methods. Construction and properties
- Examples
- Other settings and related methods
- Ongoing and future work

Motivation

Systems of ODEs and DAEs that model electronic circuits:

$$W\mathbf{y}'(t) = \mathbf{h}(\mathbf{y}(t)) + g_\omega(t)\mathbf{f}(\mathbf{y}(t)), \qquad \mathbf{y}(0) = \mathbf{y}_0.$$

We will assume:

1. W is a constant matrix:
    - If W is non-singular $\to$ system of ODEs.
    - If W is singular $\to$ system of DAEs.
2. The functions $\mathbf{h}$ and $\mathbf{f}$ are smooth and do not depend on ω.The function $\mathbf{h}$ can usually be split into a linear and a nonlinear part:

$$\mathbf{h}(\mathbf{y}(t)) = A\mathbf{y}(t) + \mathbf{m}(\mathbf{y}(t)).$$

3. The forcing term $g_\omega(t)$ is (highly) oscillatory.

Forcing terms

The term $g_\omega(t)$ can have:

- One frequency (amplitude modulation):

$$g_\omega(t) = \mathrm{e}^{\mathrm{i}\omega t}, \qquad g_\omega(t) = \cos\omega t \qquad g_\omega(t) = \sin\omega t, \qquad \omega \gg 1.$$

- Two frequencies (Double-Sided Suppressed Carrier):

$$g_\omega(t) = \sin\omega_1 t \, \sin\omega_2 t, \qquad \omega_1 \gg \omega_2 \gg 1.$$

- Full spectrum (equations for diodes and transistors):

$$g_\omega(t) = \mathrm{e}^{\eta\cos\omega t}, \qquad \omega \gg 1.$$

A linear example

Consider the following (linear) differential equation:

$$y''(t) + y(t) = 2\sin\omega t, \quad y(0) = 1, \quad y'(0) = 0.$$

or if $\mathbf{y}(t) = [y(t)\ y'(t)]^T$,

$$\mathbf{y}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y}(t) + \sin\omega t \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Set $\omega = 10^4$ and solve it with Matlab ode45 routine.

An example

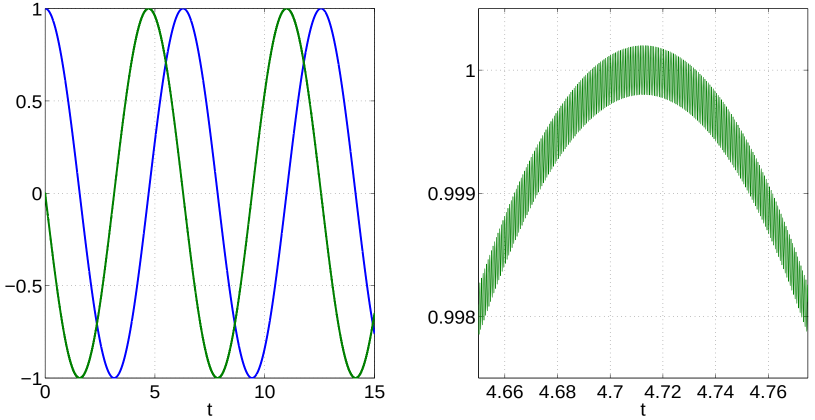

Figure: Plot of $f(t)$ (blue) and $f'(t)$ (green) given by Matlab with relative tolerance of 10^{-8} (left). Detailed plot of $f'(t)$, showing rapid $\mathcal{O}(\omega^{-1})$ oscillation (right).

Some ideas

Some ideas that follow from this and other examples are:

- The solutions seem to be smooth base functions superimposed with rapid oscillations that decrease in amplitude when ω grows.
- As a consequence, standard ODE solvers become less and less efficient when ω is large.

The form of the exact solution

Given the system

$$\mathbf{y}'(t) = \mathbf{h}(\mathbf{y}(t)) + g_\omega(t)\mathbf{f}(\mathbf{y}(t)), \qquad \mathbf{y}(0) = \mathbf{y}_0,$$

if we see it as a perturbed form of

$$\mathbf{z}'(t) = \mathbf{h}(\mathbf{z}(t)), \qquad \mathbf{z}(0) = \mathbf{y}_0,$$

then non-linear variation of constants (or Alekseev-Gröbner) gives

$$\mathbf{y}(t) = \mathbf{z}(t) + \int_0^t \mathbf{\Phi}(t-s)\mathbf{f}(\mathbf{y}(s))g_\omega(s)ds,$$

where $\mathbf{\Phi}$ solves the so-called variational equation

$$\mathbf{\Phi}' = \frac{\partial \mathbf{h}(\mathbf{z}(t))}{\partial \mathbf{z}}\mathbf{\Phi}, \qquad \mathbf{\Phi}(0) = I.$$

The form of the exact solution

In the linear case

$$\begin{aligned}\mathbf{y}(t) &= \mathbf{z}(t) + \int_0^t \mathrm{e}^{(t-x)A}\mathbf{f}(\mathbf{y}(x))g_\omega(x)\mathrm{d}x \\ &= \mathrm{e}^{At}\mathbf{y}_0 + \int_0^t \mathrm{e}^{(t-x)A}\mathbf{f}(x,\mathbf{y}(x))g_\omega(x)\mathrm{d}x.\end{aligned}$$

The matrix $\mathbf{\Phi}$ may not be analytically available in general, but if $g_\omega(t)$ is a trigonometric function, then

$$\int_0^t \mathbf{\Phi}(t-x)\mathbf{f}(\mathbf{y}(x))g_\omega(x)\mathrm{d}x = \mathcal{O}(\omega^{-1}), \qquad \omega \to \infty.$$

Instead of approximating this integral using standard quadrature, we apply the ideas from highly oscillatory problems.

Oscillatory integrals

For Fourier-type integrals

$$I[f] = \int_a^b f(x)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x,$$

we can integrate by parts (if $f(x)$ is smooth and $g'(x) \neq 0$) to get an *asymptotic expansion* in inverse powers of ω:

$$I[f] \sim -\sum_{k=0}^{\infty} \frac{1}{(-\mathrm{i}\omega)^{k+1}} \left[\frac{\sigma_k(b)}{g'(b)}\mathrm{e}^{\mathrm{i}\omega g(b)} - \frac{\sigma_k(a)}{g'(a)}\mathrm{e}^{\mathrm{i}\omega g(a)} \right],$$

where

$$\sigma_0(x) = f(x), \quad \sigma_k(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sigma_{k-1}(x)}{g'(x)}, \quad k \geq 1$$

Note that we only need information about $f(x)$ and $g(x)$ at the endpoints.

Filon-type methods

Filon-type methods are based on asymptotic ideas plus Hermite-type interpolation (Filon, Flinn, Iserles and Nørsett).

Theorem

Given $g(x)$ such that $g'(x) \neq 0$ in $[a,b]$, and $f \in \mathcal{C}^\infty[a,b]$, if we interpolate $f(x)$ by a polynomial $p(x)$ in a Hermite sense,

$$p^{(j)}(x_k) = f^{(j)}(x_k), \qquad j = 0, 1, \dots m_k - 1, \qquad k = 1, 2, \dots \nu,$$

then the Filon quadrature $Q[f] = \int_a^b p(x) e^{i\omega g(x)} dx$ satisfies

$$E[f] = \int_a^b f(x) e^{i\omega g(x)} dx - Q[f] = \mathcal{O}(\omega^{-s-1}),$$

where $s = \min\{m_1, m_\nu\}$.

In particular, if we interpolate at the endpoints,

$$p(a) = f(a), \qquad p(b) = f(b),$$

then $E[f] = \mathcal{O}(\omega^{-2})$.

Consider the previous example,

$$\mathbf{y}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y}(t) + \sin \omega t \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

with $\mathbf{y}(0) = [1, 0]^T$.

Filon-type methods. Example

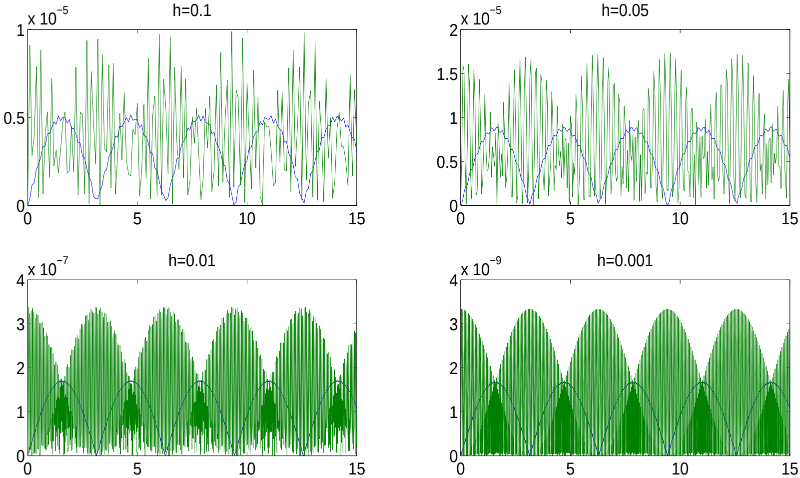

Figure: Errors when solving the previous ODE with simple Filon and different stepsizes. Here $\omega = 100$.

Filon-type methods. Example

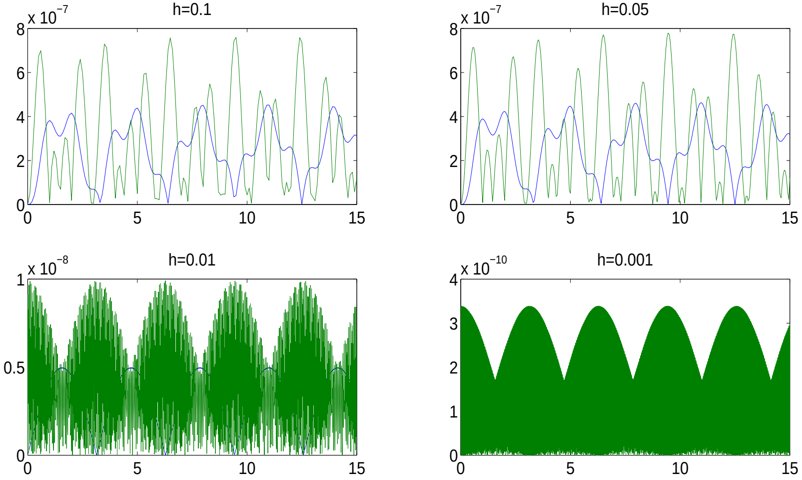

Figure: Errors when solving the previous ODE with simple Filon and different stepsizes. Here $\omega = 1000$.

Modulated Fourier series

All this motivates the *ansatz*:

$$\mathbf{y}(t) = \sum_{n=0}^{\infty} \frac{\psi_n(t)}{\omega^n},$$

where the functions $\psi_n(t)$ can be expanded in modulated Fourier series:

$$\psi_n(t) = \sum_{m=-\infty}^{\infty} \mathbf{p}_{n,m}(t) \mathrm{e}^{\mathrm{i}m\omega t}, \qquad n \geq 0.$$

This procedure has two main advantages:

- Large values of ω will be beneficial, because we will need fewer terms for a good approximation.
- Small time-stepping can be avoided.

Computation of the asymptotic expansion

The standard procedure is as follows:

- We expand everything (formally) in inverse powers of ω using the ansatz.
- This normally involves
  - a separation of orders of magnitude (powers of ω),
  - a separation of frequencies (values of m).
- Typically one obtains either *nonoscillatory* ODEs or recursions for the coefficients $\mathbf{p}_{n,m}(t)$.
- Note that we do NOT solve any oscillatory ODE using standard methods! The only point where oscillatory elements come in is when assembling the terms $\psi_n(t)$.

The Fourier oscillator

As an example, consider

$$\mathbf{y}'(t) = \mathbf{h}(\mathbf{y}(t)) + g_\omega(t)\mathbf{f}(\mathbf{y}(t)), \qquad \mathbf{y}(0) = \mathbf{y}_0,$$

where

$$g_\omega(t) = \sum_{m=-\infty}^{\infty} \alpha_m \mathrm{e}^{\mathrm{i}m\omega t},$$

and $\mathbf{h}$ and $\mathbf{f}$ are smooth.

If we equate $\mathcal{O}(1)$ terms, we have

$$\mathbf{p}_{0,0}'(t) = \mathbf{h}(\mathbf{p}_{0,0}(t)) + \alpha_0 \mathbf{f}(\mathbf{p}_{0,0}(t)), \qquad \mathbf{p}_{0,0}(0) = \mathbf{y}(0) = \mathbf{y}_0,$$

which is nonoscillatory. Additionally

$$\mathbf{p}_{1,m}(t) = \frac{\alpha_m}{\mathrm{i}m} \mathbf{f}(\mathbf{p}_{0,0}(t)), \qquad m \neq 0.$$

The first term

The $\mathcal{O}(\omega^{-1})$ level yields again a differential equation for $\mathbf{p}_{1,0}$:

$$\mathbf{p}_{1,0}' = \mathbf{Jh}(\mathbf{p}_{0,0})\mathbf{p}_{1,0} + \mathbf{Jf}(\mathbf{p}_{0,0}) \sum_{r=-\infty}^{\infty} \alpha_r \, \mathbf{p}_{1,-r},$$

together with $\mathbf{p}_{1,0}(0) = \mathbf{0}$, and a recursion for the next level:

$$\mathbf{p}_{2,m} = -\frac{\mathrm{i}}{m} \left[-\mathbf{p}_{1,m}' + \mathbf{Jh}(\mathbf{p}_{0,0})\mathbf{p}_{1,m} + \mathbf{Jf}(\mathbf{p}_{0,0}) \sum_{r=-\infty}^{\infty} \alpha_r \mathbf{p}_{1,m-r} \right],$$

for $m \neq 0$.

The pattern

Each level $n \geq 0$ will provide the following information:

- A *nonoscillatory* ODE for $\mathbf{p}_{n,0}(t)$.
- The coefficients $\mathbf{p}_{n+1,m}(t)$ for $m \neq 0$.
- This in turn gives the initial conditions for the ODE corresponding to the next level, $\mathbf{p}_{n+1,0}(t)$, imposing that

$$\psi_0(0) = \mathbf{y}_0, \qquad \psi_n(0) = \mathbf{0}, \qquad n \geq 1,$$

so

$$\mathbf{p}_{0,0}(0) = \mathbf{y}_0, \qquad \mathbf{p}_{n,0}(0) = -\sum_{m \neq 0} \mathbf{p}_{n,m}(0), \qquad n \geq 1,$$

- Then we only need to assemble the modulated Fourier expansion...

Band limited input and blossoming

All the computations can be simplified if the forcing term is *band limited*, that is

$$g_\omega(t) = \sum_{m=-\infty}^{\infty} a_m(t)\mathrm{e}^{\mathrm{i}m\omega t},$$

and there exists ϱ such that $a_m \equiv 0$ if $|m| \geq \varrho + 1$.

The first $\psi_n(t)$ preserve this bandwidth, but higher order terms show an increase in the number of nonzero frequencies. This rate of increase (analysis and prediction of intermodulation distortion in Engineering) can be estimated for general systems from the recurrence relations.

Band limited input and blossoming

Theorem

For $s \geq 0$, the maximum bandwidth θ_s of the term $\psi_s(t)$ is

$$\theta_s = s\varrho,$$

where ϱ is the bandwidth of the original forcing term.

Theorem

Let $\mathbf{f}(\mathbf{y})$ be constant, then we have $\theta_0 = 0$, $\theta_1 = \varrho$ and the maximum bandwidth θ_s of the term $\psi_s(t)$ is

$$\theta_s = (s-1)\varrho, \qquad s \geq 2.$$

An example

Consider the following nonlinear example, given by the following system (Bartuccelli, Deane, Gentile):

$$C\frac{dV_C}{dt} = I_L + f(V_C), \qquad L\frac{dI_L}{dt} = -RI_L - V_C,$$

where

$$f(V_C) = AV_C\left(1 - \frac{V_C^2}{V_{dd}^2}\right),$$

and A, V_{dd}, C, L and R are parameters of the system. A periodic perturbation can be introduced as follows:

$$f(V_C, t) = (A + B\sin\Omega t)V_C\left(1 - \frac{V_C}{V_{dd}^2}\right), \qquad A > 0, \quad B \in \mathbb{R}.$$

An example

After normalisation and scaling, this system can be written as

$$\frac{du}{dt} = \alpha v + \Phi(t)u(1-u^2), \qquad \frac{dv}{dt} = -u - v, \tag{1}$$

where $\Phi(t) = \beta + \mu \sin \omega t$ and

$$\alpha = \frac{L}{R^2C}, \quad \beta = \frac{LA}{RC}, \quad \mu = \frac{LB}{RC}, \quad \omega = \frac{\Omega L}{R}$$

Hence,

$$\mathbf{h}(u,v) = \begin{pmatrix} \alpha v + \beta u(1-u^2) \\ -u-v \end{pmatrix},$$

and

$$\mathbf{f}(u,v) = \begin{pmatrix} u(1-u^2) \\ 0 \end{pmatrix},$$

together with $g_\omega(t) = \mu \sin \omega t$.

An example

The first term of the expansion solves the system

$$\mathbf{p}_{0,0}' = \mathbf{h}(\mathbf{p}_{0,0}),$$

since $a_0 \equiv 0$, and additionally

$$\mathbf{p}_{1,-1} = -\frac{\mu}{2}\mathbf{f}(\mathbf{p}_{0,0}) = \mathbf{p}_{1,1},$$

together with $\mathbf{p}_{1,m} \equiv \mathbf{0}$ when $|m| > 1$. The coefficient $\mathbf{p}_{1,0}$ satisfies the ODE

$$\mathbf{p}_{1,0}' = \mathbf{J}[\mathbf{h}](\mathbf{p}_{0,0})\mathbf{p}_{1,0}, \qquad \mathbf{p}_{1,0}(0) = \mu\mathbf{f}(\mathbf{p}_{0,0}(0)).$$

Putting everything together, we have

$$\psi_1(t) = \mathbf{p}_{1,0} - \mu\mathbf{f}(\mathbf{p}_{0,0})\cos\omega t.$$

An example

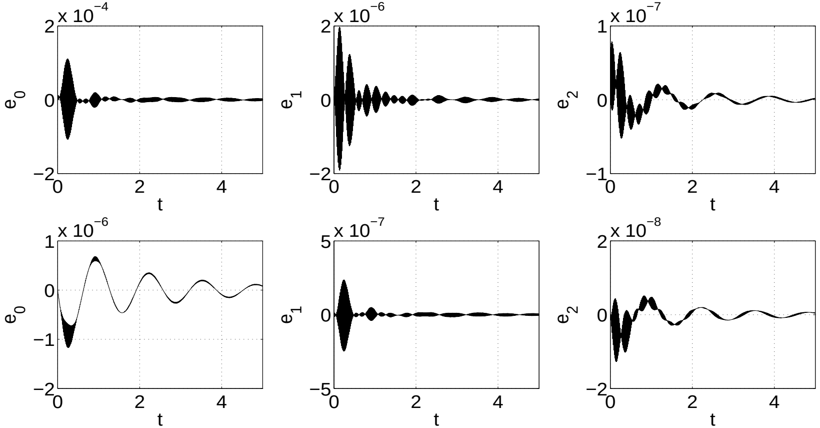

Figure: Absolute errors in the approximation of the solution of the perturbed system (1) for $C = 10^{-6}$ and $\Omega = 2\pi \times 10^6$. Top row, errors in $u(t)$ (zeroth, zeroth plus first and zeroth plus first plus second terms, from left to right). Bottom row, same for $v(t)$.

An expcos example

In the modelling of diode circuits with inductive loads, we can find an equation of the form

$$x'(t) = -\frac{L}{RC}x(t) + \frac{I_s L}{C}\left[\exp\left(\frac{g_\omega(t) - x(t)}{V_T}\right) - 1\right] - \frac{L}{C}y(t),$$
$$y'(t) = x(t),$$

where L, R, C, I_s and V_T are parameters. We will take the values $L = 10^{-4}$, $R = 100$, $C = 10^{-6}$, $I_s = 10^{-12}$ and $V_T = 0.0259$.

En expcos example

So

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} \beta e^{-x(t)/V_T} \\ 0 \end{pmatrix} \exp\left(\frac{g_\omega(t)}{V_T}\right) - \begin{pmatrix} \beta \\ 0 \end{pmatrix},$$

where $\beta = I_s L/C$, and

$$A = \begin{pmatrix} -L/RC & -L/C \\ 1 & 0 \end{pmatrix}.$$

The forcing term is $g_\omega(t) = \mu \cos \omega t$, with large ω.

An expcos example

A useful identity is

$$e^{\mu \cos \omega t} = I_0(\mu) + 2 \sum_{m=1}^{\infty} I_m(\mu) \cos m\omega t,$$

together with the large m asymptotic behaviour of the modified Bessel functions:

$$I_m(z) \sim \frac{1}{\sqrt{2\pi m}} \left(\frac{ez}{2m}\right)^m, \qquad m \to \infty.$$

Also, a uniform expansion (for $0 < z < \infty$) is

$$I_m(mz) \sim \frac{e^{m\eta}}{(2\pi m)^{\frac{1}{2}} (1+z^2)^{\frac{1}{4}}}, \qquad m \to \infty,$$

where

$$\eta = (1+z^2)^{\frac{1}{2}} + \ln \frac{z}{1+(1+z^2)^{\frac{1}{2}}}.$$

An expcos example

In this case, we obtain

$$\mathbf{p}_{0,0} = \mathbf{h}(\mathbf{p}_{0,0}) + I_0(\mu)\mathbf{f}(\mathbf{p}_{0,0}), \qquad \mathbf{p}_{0,0}(0) = \mathbf{x}(0),$$

and also

$$\mathbf{p}_{1,m} = -\frac{iI_m(\mu)}{m}\mathbf{f}(\mathbf{p}_{0,0}), \qquad m \neq 0.$$

$$\mathbf{p}_{1,0}' = \left(\mathbf{J}[\mathbf{h}] + I_0(\mu)\mathbf{J}[\mathbf{f}]\right)\mathbf{p}_{1,0}, \qquad \mathbf{p}_{1,0}(0) = \mathbf{0},$$

which implies $\mathbf{p}_{1,0} \equiv \mathbf{0}$. Therefore,

$$\psi_1(t) = 2\mathbf{f}(\mathbf{p}_{0,0}) \sum_{m=1}^{\infty} \frac{I_m(\mu)}{m} \sin m\omega t.$$

The second term $\psi_2(t)$ can be computed similarly.

More on modulated Fourier expansions

Other settings and related methods:

- Equations of the form $\ddot{x} + \Omega^2 x = g(x)$, where

$$\Omega = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \omega I \end{array} \right),$$

  and $\omega \gg 1$.
  - Hairer and Lubich (SIAM J. Num. Anal. 2000),
  - Cohen, Hairer and Lubich (FoCM, 2003),
  - Cohen (PhD thesis, 2004),
  - Hairer, Lubich and Wanner (Springer, 2006).

More on modulated Fourier expansions

- Equations with ω-dependent coefficients, like the inverted pendulum:

$$\theta''(t) = l^{-1}(g + \sigma\omega\cos\omega t)\sin\theta(t),$$

  with $\theta(0) = \theta_0$, $\theta'(0) = \theta_0'$.

  See also E and Engquist (2003), Sanz–Serna (2009).
- The construction is closely related to stroboscopic and higher order averaging:
  - Calvo, Chartier, Murua, Sanz-Serna (2010),
  - Chartier, Murua, Sanz-Serna (2011).
  - Several talks last Monday...

To sum up

We have presented an asymptotic-numerical method to solve efficiently highly oscillatory systems of ODEs, based on:

- Asymptotic expansions in inverse powers of the oscillatory parameter ω.
- Modulated Fourier expansions.
- Solving *non-oscillatory* ODEs and recursions for the coefficients in the expansion.
- Computing effort which is essentially independent of ω.
- DAEs, delay differential equations in progress...

And finally...

That's all for now...

Thank you for your attention!