Quasi-stroboscopic averaging: The non-autonomous case

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Outline

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Consider highly-oscillatory problems (HOPs) with:

- a clear and explicit separation of the time-scales
- a quasi-periodic dependence in the fast time

Highly-oscillatory problems with periodic time-dependence

$$\left\{\begin{array}{rcl} y'(t) & = & \varepsilon f(y(t), t\omega) \in \mathbb{R}^D, \quad \omega \in \mathbb{R}^d, \\ y(0) & = & y_0 \end{array}\right.$$

where :

- ε is a small parameter (scales as the inverse of the frequency).
- $f(y,\theta)$ is a **smooth**, 2π-periodic w.r.t. to each angle θ_i, and possesses a Fourier expansion $f(y,\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^\mathbf{d}} e^{i(\mathbf{k}\cdot\theta)} f_\mathbf{k}(y)$.

Examples of non-autonomous multi-frequency problems:

- Mathieu's equation:

$$\ddot{x} + \omega_0^2(1 - \varepsilon\cos(t))x = 0.$$

- Duffing oscillator:

$$\ddot{x} + \beta\dot{x} + \omega_0^2 x + \alpha x^3 = \varepsilon\omega_0^2\cos(t)$$

- A class of non-autonomous systems (will be treated in full generality in Ander's talk):

$$\dot{x} = Ax + \varepsilon h(x)$$

rewritten as

$$\dot{z} = \varepsilon e^{-tA} h\left(e^{tA} z\right)$$

$\rightarrow$ Fermi-Pasta-Ulam type problem.

Outline

Appears in various forms:

- “Standard” form in KAM theory
- Two-scale expansions in quantum mechanics (Wentzel-Kramers-Brillouin, 1926)
- Magnus expansions (A. Iserles, 2002)
- Modulated Fourier expansions (E. Hairer and Chr. Lubich, 2000)

Theory has been gradually improved for the systems considered here:

- Krylov and Bogoliubov (1934) : basic idea
- Bogoliubov and Mitropolski (1958) : rigorous statement for second order approximation and general scheme
- Perko (1969) : almost complete theory with error estimates for the periodic and quasi-periodic cases (see also the book of Sanders, Verhulst and Murdock, 2007)

Theorem (Perko)

For a smooth $f : \mathbb{R}^D \times \mathbb{T}^d \to \mathbb{R}^D$ *and a vector* $\omega \in \mathbb{R}^d$ *such that for all* $\mathbf{k} \in \mathbb{Z}^d$, $|\mathbf{k} \cdot \omega| \geq c|k|^{-\nu}$, *consider the ODE*

$$y'(t) = \varepsilon f(y(t), t\omega), \quad y(0) = y_0.$$

Then for any $N > 1$, *there exist:*

- *(i) a map* $U_N = id + \ldots + \varepsilon^{N-1} u_{N-1}$ *from* $\mathbb{R}^D \times \mathbb{T}^d$ *to* $\mathbb{R}^D$,
- *(ii) a vector field* $\varepsilon F_1 + \ldots + \varepsilon^N F_N$,
- *(iii) a constant* C_N,

such that the solution Y *of the autonomous ODE*

$$Y' = \varepsilon F_1(Y) + \ldots + \varepsilon^N F_N(Y), \quad U_N(Y(0), \mathbf{0}) = y_0,$$

satisfies

$$\|y(t) - U_N(Y(t), t\omega)\| \leq C_N \varepsilon^N \text{ for } t \leq L/\varepsilon.$$

The functions u_i (and thus F_i) are **not unique** except for F_1

$$\tilde{F}_1(Y) = f(y,\theta),$$

$$\tilde{F}_j(Y,\theta) = \sum_{k=1}^{j-1}\left[\frac{1}{k!}\sum_{i_1+\ldots+i_k=j-1}\frac{\partial^k f}{\partial y^k}\left(u_{i_1},\ldots,u_{i_k}\right) - \frac{\partial u_k}{\partial Y}F_{j-k}\right],$$

$$F_j(Y) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\tilde{F}_j(Y,\theta)d\theta,$$

$$\omega\cdot\frac{\partial u_j}{\partial\theta}(Y,\theta) = \tilde{F}_j(Y,\theta) - F_j(Y).$$

- $d = 1$: We can impose $u_j(Y,0) = 0$, this is **stroboscopic** or averaging, in the sense that $U(Y,2k\pi) = Y$;
- $d > 1$: The choice $\int_{\mathbb{T}^d} u_j(Y,\theta)d\theta = 0$ is the **one prescribed** in Perko's theorem.

Why do we consider stroboscopic averaging using B-series rather than anything else?

- Quasi-stroboscopic averaging is intrinsically geometric.
- B-series allow the derivation of fully explicit expansions, a very useful feature for the analysis of numerical methods.
- Stroboscopic averaging has proved to be the portal for new numerical schemes in the mono-frequency situation and it is expected that methods can be developed based on quasi-stroboscopic averaging.

Outline

The exact oscillatory solution is obtained in three steps:

1. **First step: Integral formulation**

$$y(t) = y_0 + \varepsilon \int_0^t f(y(s), s\omega) ds$$

2. **Second step: Fourier expansion of the vector field**

$$y(t) = y_0 + \varepsilon \int_0^t \sum_{\mathbf{k} \in \mathbb{Z}^d} e^{is(\mathbf{k}\cdot\omega)} f_{\mathbf{k}}(y(s)) ds$$

3. **Third step: insertion *à la Picard* and Taylor expansion**

$$\begin{aligned} y(t) &= \cdots + \varepsilon \sum_{\mathbf{k}} \left(\int_0^t e^{is(\mathbf{k}\cdot\omega)} ds \right) f_{\mathbf{k}}(y_0) + \mathcal{O}(\varepsilon^2) \\ &= \cdots + \varepsilon^2 \sum_{\mathbf{k},\mathbf{l}} \left(\int_0^t \int_0^{s_1} e^{is_1(\mathbf{k}\cdot\omega) + s_2(\mathbf{l}\cdot\omega)} ds_1 ds_2 \right) (f'_{\mathbf{k}} f_{\mathbf{l}})(y_0) + \mathcal{O}(\varepsilon^3) \\ &\vdots \end{aligned}$$

The procedure may be pursued iteratively with increasing complexity. It involves two ingredients of different nature:

1. **Problem-dependent elements**: so-called *elementary differentials* based on the Fourier coefficients of f at y_0

$$f_{\mathbf{k}}(y_0), \quad f'_{\mathbf{k}}(y_0)f_{\mathbf{l}}(y_0), \quad f''_{\mathbf{k}}(y_0)\left(f_{\mathbf{l}}(y_0), f_{\mathbf{m}}(y_0)\right), \cdots$$

2. **Universal elements**: time-dependent coefficients

$$\int_0^t e^{is(\mathbf{k}\cdot\omega)}ds, \quad \int_0^t\int_0^{s_1} e^{is_1(\mathbf{k}\cdot\omega)+s_2(\mathbf{l}\cdot\omega)}ds_1ds_2, \quad \cdots$$

depending only on ω

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The set $\mathcal{T}$ of *mode-coloured* trees is defined recursively by:

1. For all $\mathbf{k} \in \mathbb{Z}^d$, $\textcircled{\scriptsize k}$ belongs to $\mathcal{T}$;
2. If $u_1, \ldots, u_n$ are n trees of $\mathcal{T}$, then, the tree

$$u = [u_1, \ldots, u_n]_{\mathbf{k}}$$

   obtained by connecting their roots to a new root with multi-index $\mathbf{k} \in \mathbb{Z}^d$, belongs to $\mathcal{T}$.

The order $|u|$ of a tree $u \in \mathcal{T}$ is its number of nodes.

Elementary differentials are defined recursively by the formulae:

1. $\mathcal{F}_{\textcircled{\scriptsize k}}(y_0) = f_{\mathbf{k}}(y_0)$
2. $\mathcal{F}_{[u_1,\dots,u_n]_{\mathbf{k}}}(y_0) = \frac{\partial^n f_{\mathbf{k}}}{\partial y^n}(y_0)\Big(\mathcal{F}_{u_1}(y_0),\dots,\mathcal{F}_{u_n}(y_0)\Big)$

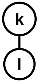
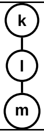
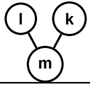
u				
$\lvert u \rvert$	1	2	3	3
$\mathcal{F}_u(y)$	$f_{\mathbf{k}}(y)$	$f'_{\mathbf{l}}(y)f_{\mathbf{k}}(y)$	$f'_{\mathbf{m}}(y)f'_{\mathbf{l}}(y)f_{\mathbf{k}}(y)$	$f''_{\mathbf{m}}(y)(f_{\mathbf{l}}(y),f_{\mathbf{k}}(y))$

Figure: Trees of orders ≤ 3 and associated elementary differentials

Mode-coloured B-series are power series indexed by trees:

$$B(\delta, y) = \delta_\emptyset y + \sum_{u \in \mathcal{T}} \frac{\varepsilon^{|u|}}{\sigma_u} \delta_u \mathcal{F}_u(y)$$

where σ is the symmetry factor and $\alpha \in \mathbb{C}^{\mathcal{T} \cup \{\emptyset\}}$.

Pluging $y(t) = B(\alpha(t), y_0)$ into the integral formulation leads to:

$$\forall \mathbf{k} \in \mathbb{Z}^d, \quad \alpha_{\textcircled{\scriptsize k}}(t) = \int_0^t e^{is\mathbf{k}\cdot\omega} ds$$

$$\forall u = [u_1, \ldots, u_n]_{\mathbf{k}}, \quad \alpha_u(t) = \int_0^t e^{is(\mathbf{k}\cdot\omega)} \alpha_{u_1}(s) \cdots \alpha_{u_n}(s) ds.$$

- Given two B-series $B(\delta, y)$ with $\delta_\emptyset = 1$ and $B(\eta, y)$, their composition

$$B(\eta, B(\delta, y))$$

is a B-series with coefficients $\alpha * \beta \in \mathbb{C}^{\mathcal{T} \cup \{\emptyset\}}$.

- This law endows

$$\mathcal{G} := \{\delta \in \mathbb{C}^{\mathcal{T} \cup \{\emptyset\}} \ : \ \delta_\emptyset = 1\}$$

with a Lie-group structure, with neutral element $\mathbb{1}$.

- The corresponding Lie-algebra is

$$\begin{aligned} \mathfrak{g} &= \{\beta \in \mathbb{C}^{\mathcal{T} \cup \{\emptyset\}} : \beta_\emptyset = 0\} \\ &= \{ \left. \frac{d\alpha(t)}{dt} \right|_{t=0} \text{ for } \alpha(t) \in \mathcal{G} \text{ with } \alpha(0) = \mathbb{1}\} \end{aligned}$$

Our first task is to rewrite the vector field itself as a B-series

$$\varepsilon \sum_{\mathbf{k}\in\mathbb{Z}^d} e^{i(\mathbf{k}\cdot\theta)} f_k(y) = B(\beta(\theta), y) = \sum_{u\in\mathcal{T}} \frac{\varepsilon^{|u|}}{\sigma_u} \beta_u(\theta)\mathcal{F}_u(y)$$

with coefficients $\beta_u(\theta)$ defined for $u \in \mathcal{T} \cup \{\emptyset\}$ as follows:

$$\beta_u(\theta) = \begin{cases} e^{i(\mathbf{k}\cdot\theta)} & \text{if } u = \textcircled{\scriptsize k} \text{ for some } \mathbf{k} \in \mathbb{Z}^d, \\ 0 & \text{otherwise.} \end{cases}$$

Writing the IVP in terms of B-series, we obtain

$$\begin{aligned} \frac{d}{dt} B(\alpha(t), y_0) &= B(\alpha(t) * \beta(t\omega)), y_0), \\ B(\alpha(0), y_0) &= B(1\!\!1, y_0), \end{aligned}$$

Consequence: α is a curve in $\mathcal{G}$ satisfying the IVP

$$\begin{cases} \frac{d}{dt}\alpha(t) & = & \alpha(t) * \beta(t\omega), \\ \alpha(0) & = & \mathbb{1} \end{cases}$$

Since $\beta_u(\theta) = 0$ whenever $|u| \neq 1$, we obtain for $u = [u_1 \cdots u_n]_{\mathbf{k}}$

$$\frac{d\alpha_u(t)}{dt} = \beta_{\textcircled{\scriptsize k}}(t\omega)\alpha_{u_1}(t)\ldots\alpha_{u_n}(t)$$

and after integration

$$\alpha_u(t) = \int_0^t e^{is(\mathbf{k}\cdot\omega)}\alpha_{u_1}(s)\cdots\alpha_{u_n}(s)ds.$$

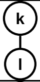
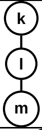
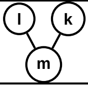
u	$\alpha_u(t)$
	$\int_0^t e^{is_1(\mathbf{k}\cdot\omega)}\, ds_1$
	$\int_0^t \int_0^{s_1} e^{i(s_1\mathbf{k}+s_2\mathbf{l})\cdot\omega}\, ds_1\, ds_2$
	$\int_0^t \int_0^{s_1} \int_0^{s_2} e^{i(s_1\mathbf{k}+s_2\mathbf{l}+s_3\mathbf{m})\cdot\omega}\, ds_1\, ds_2\, ds_3$
	$\int_0^t e^{is_1(\mathbf{m}\cdot\omega)} \left(\int_0^{s_1} e^{is_2(\mathbf{l}\cdot\omega)} ds_2 \int_0^{s_1} e^{is_3(\mathbf{k}\cdot\omega)} ds_3\right) ds_1$

Figure: First coefficients of the oscillatory B-series expansion

Outline

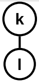
Lemma

Assume that ω is non-resonant and consider $w \in \mathbb{C}^{\mathbb{R}\times\mathbb{T}^d}$ a continuous function, such that for all θ, $w(\cdot,\theta)$ is polynomial. If for all t, $w(t,t\omega)=0$, then for all (t,θ), $w(t,\theta)=0$.

Consequence: for all $u \in \mathcal{T}$, $\alpha_u(t)$ is of the form

$$\alpha_u(t) = P_u(t, e^{it\omega_1}, \dots, e^{it\omega_d}, e^{-it\omega_1}, \dots, e^{-it\omega_d})$$

where $P_u \in \mathbb{C}[X, Z_1, \dots, Z_d, Z_1^{-1}, \dots, Z_d^{-1}]$ is defined **uniquely**.

Consider $u =$ 

with $\mathbf{l} \neq -\mathbf{k}$, $\mathbf{k}, \mathbf{l} \neq 0$: then

$$\alpha_u = \frac{-(\mathbf{k}\cdot\omega) + ((\mathbf{l}+\mathbf{k})\cdot\omega)e^{it(\mathbf{l}\cdot\omega)} - (\mathbf{l}\cdot\omega)e^{it((\mathbf{l}+\mathbf{k})\cdot\omega)}}{(\mathbf{k}\cdot\omega)(\mathbf{l}\cdot\omega)\left((\mathbf{l}+\mathbf{k})\cdot\omega\right)}$$

$$P_u = \frac{-(\mathbf{k}\cdot\omega) + ((\mathbf{l}+\mathbf{k})\cdot\omega)\prod_{i=1}^d Z_i^{\mathbf{l}_i} - (\mathbf{l}\cdot\omega)\prod_{i=1}^d Z_i^{\mathbf{l}_i+\mathbf{k}_i}}{(\mathbf{k}\cdot\omega)(\mathbf{l}\cdot\omega)\left((\mathbf{l}+\mathbf{k})\cdot\omega\right)}$$

We can finally write $\alpha_u(t) = \gamma_u(t,\theta)|_{\theta=t\omega}$ where

$$\gamma_u(t,\theta) = P_u(t, e^{i\theta_1}, \ldots, e^{i\theta_d}, e^{-i\theta_1}, \ldots, e^{-i\theta_d})$$

is a function of $\mathbb{C}^{\mathbb{R}\times\mathbb{T}^d}$.

The chain rule and the density argument then show that:

For all $(t,\theta) \in \mathbb{R} \times \mathbb{T}^d$

$$\begin{aligned} \partial_t \gamma(t,\theta) + \omega \cdot \nabla_\theta \gamma(t,\theta) &= \gamma(t,\theta) * \beta(\theta) \\ \gamma(0,\mathbf{0}) &= \mathbb{1} \end{aligned}$$

In full generality, for any function $\chi(\theta) \in \mathcal{G}^{\mathbb{T}^d}$ such that $\chi(\mathbf{0}) = \mathbb{1}$,

$$\gamma(t,\theta) = \chi(\theta - t\omega) + \int_0^t \gamma(s, \theta + (s-t)\omega) * \beta(\theta + (s-t)\omega) ds$$

is solution of the transport equation.

Definition

A coefficient map $\delta \in \mathcal{G}^{\mathbb{R}\times\mathbb{T}^d}$ is said to be polynomial if

$$\forall u \in \mathcal{T}, \quad \delta_u(t,\theta) = P_u(t, e^{i\theta_1}, \ldots, e^{i\theta_d}, e^{-i\theta_1}, \ldots, e^{-i\theta_d})$$

with $P_u \in \mathbb{C}[Z_0, Z_1, \ldots, Z_d, Z_1^{-1}, \ldots, Z_d^{-1}]$.

Theorem

There exists a unique polynomial solution $\gamma \in \mathcal{G}^{\mathbb{R}\times\mathbb{T}^d}$ of

$$\begin{aligned} \partial_t \gamma(t,\theta) + \omega \cdot \nabla_\theta \gamma(t,\theta) &= \gamma(t,\theta) * \beta(\theta) \\ \gamma(0,\mathbf{0}) &= \mathbb{1} \end{aligned}$$

Theorem

For all $t, t' \in \mathbb{R}$ and all $\theta \in \mathbb{T}^d$,

$$\gamma(t+t',\theta) = \gamma(t',0) * \gamma(t,\theta).$$

Proof: Let t' be fixed. By right-linearity and associativity of the convolution product $*$:

$$\begin{aligned} &(\partial_t + \omega\cdot\partial_\theta) \left(\gamma(t',\mathbf{0})^{-1} * \gamma(t'+t,\theta)\right) = \\ \gamma(t',\mathbf{0})^{-1} * \; &(\partial_t + \omega\cdot\partial_\theta) \; \gamma(t+t',\theta) \\ \gamma(t',\mathbf{0})^{-1} * \; &\gamma(t'+t,\theta) \; * \; \beta(\theta) \end{aligned}$$

Hence, $\gamma(t',\mathbf{0})^{-1} * \gamma(t'+t,\theta)$ satisfies the transport equation with

$$\gamma(t',\mathbf{0})^{-1} * \gamma(t',\mathbf{0}) = \mathbb{1}$$

.

Outline

Reminder: $\gamma(t+t',\theta) = \gamma(t',0) * \gamma(t,\theta)$

- Averaged solution: $\bar{\alpha} \in \mathcal{G}^{\mathbb{R}}$ is the coefficient map obtained by freezing the oscillations in α:

$$\bar{\alpha}(t) = \gamma(t, \mathbf{0})$$

  Previous theorem shows $\bar{\alpha}(t)$ is a 1-parameter group.
- Averaged vector field: By standard results on ODEs, $\bar{\alpha}(t)$ satisfies the *autonomous* ODE

$$\frac{d}{dt}\bar{\alpha}(t) = \bar{\alpha}(t) * \bar{\beta} \quad \text{with} \quad \bar{\beta} = \frac{d}{dt}\bar{\alpha}(t)\Big|_{t=0} = \frac{\partial}{\partial t}\gamma(t,\mathbf{0})\Big|_{t=0}$$

- Change of variables: Previous theorem then shows that

$$\alpha(t) = \bar{\alpha}(t) * \kappa(t\omega) \quad \text{with} \quad \kappa(\theta) = \gamma(0,\theta)$$

Quasi-stroboscopic averaging: a schematic view

$$\left\{\begin{array}{l}\frac{dy(t)}{dt}=\varepsilon f(y(t),t\omega),\\ y(0)=y_0\end{array}\right. \quad \xleftrightarrow[\varepsilon f(y,\theta)=B(\beta(\theta),y)]{y(t)=B(\alpha(t),y_0)} \quad \left\{\begin{array}{l}\frac{d\alpha(t)}{dt}=\alpha(t)*\beta(t\omega),\\ \alpha(0)=\mathbb{1}\end{array}\right.$$

$$y(t)=U(Y(t),t\omega) \quad \updownarrow \qquad\qquad \updownarrow \quad \alpha(t)=\bar{\alpha}(t)*\kappa(t\omega)$$

$$\left\{\begin{array}{l}\frac{dY(t)}{dt}=\varepsilon F(Y(t)),\\ Y(0)=Y_0=y_0\end{array}\right. \quad \xleftrightarrow[\varepsilon F(Y)=B(\bar{\beta},Y)]{Y(t)=B(\bar{\alpha}(t),Y_0)} \quad \left\{\begin{array}{l}\frac{d\bar{\alpha}(t)}{dt}=\bar{\alpha}(t)*\bar{\beta},\\ \bar{\alpha}(0)=\mathbb{1}\end{array}\right.$$

Figure: Quasi-stroboscopic averaging in terms of B-series

Theorem

The solution of

$$y'(t) = \varepsilon f(y(t), t\omega), \quad y(0) = y_0$$

may be written as $y(t) = U(Y(t), t\omega)$ *where*

$$U(Y, \theta) = Y + \sum_{u \in \mathcal{T}} \frac{\varepsilon^{|u|}}{\sigma_u} \kappa_u(\theta) \mathcal{F}_u(Y)$$

and $Y(t)$ *is the solution of the (averaged) autonomous IVP*

$$Y'(t) = \varepsilon F(Y), \quad Y(0) = y_0$$

with

$$\varepsilon F(Y) := B(\bar{\beta}, Y) = \sum_{u \in \mathcal{T}} \frac{\varepsilon^{|u|}}{\sigma_u} \bar{\beta}_u \mathcal{F}_u(Y)$$

Outline

Consider the subgroup $\hat{\mathcal{G}} \subset \mathcal{G}$ of coefficient maps δ such that

$$\forall (u, v, w) \in \mathcal{T}^3, \qquad \begin{aligned} &\delta_{u \circ v} + \delta_{v \circ u} = \delta_u \delta_v \\ &\delta_{(u \circ v) \circ w} + \delta_{(v \circ u) \circ w} + \delta_{(w \circ u) \circ v} = \delta_u \delta_v \delta_w \end{aligned}$$

where $\circ$ denotes the Butcher product (grafting).

Remark

$\hat{\mathcal{G}}$ is the group of flows that preserve cubic polynomial invariants and/or the group of volume-preserving flows.

We denote the corresponding Lie subalgebra $\hat{\mathfrak{g}} \subset \mathfrak{g}$.

Previous relations allow the rewriting of B-series as series indexed by *words* $\mathbf{k}_1 \cdots \mathbf{k}_r$ where the $\mathbf{k}_i \in \mathbb{Z}^d$:

$$\forall \delta \in \hat{\mathcal{G}}, \quad B(\delta, y) = y + \sum_{r=1}^{\infty} \varepsilon^r \sum_{\mathbf{k}_1,\dots,\mathbf{k}_r \in \mathbb{Z}^d} \delta_{\mathbf{k}_1 \cdots \mathbf{k}_r} f_{\mathbf{k}_1 \cdots \mathbf{k}_r}$$

where:

- $\delta_{\mathbf{k}_1 \cdots \mathbf{k}_r} = \delta_{u_{\mathbf{k}_1 \cdots \mathbf{k}_r}}$ if $u_{\mathbf{k}_1 \cdots \mathbf{k}_r}$ is defined by

$$u_{\mathbf{k}_1} = \textcircled{\mathbf{k}_1} \qquad u_{\mathbf{k}_1 \cdots \mathbf{k}_r} = [u_{\mathbf{k}_1 \cdots \mathbf{k}_{r-1}}]_{\mathbf{k}_r}$$

- $f_{\mathbf{k}_1 \cdots \mathbf{k}_r}$ is defined by

$$f_{\mathbf{k}_1 \cdots \mathbf{k}_r}(y) = \partial_y f_{\mathbf{k}_1 \cdots \mathbf{k}_{r-1}}(y) f_{\mathbf{k}_r}(y)$$

The Dynkin-Specht-Wever theorem finally enables to write:

$$\forall \beta \in \hat{\mathfrak{g}}, B(\beta, y) = \sum_{r=1}^{\infty} \frac{\varepsilon^r}{r} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_r \in \mathbb{Z}^d} \beta_{\mathbf{k}_1 \cdots \mathbf{k}_r} [[\ldots[f_{\mathbf{k}_1}, f_{\mathbf{k}_2}], f_{\mathbf{k}_3}], \ldots], f_{\mathbf{k}_r}](y)$$

where, for any two vector fields g and h

$$[g, h](y) = h'(y)g(y) - g'(y)h(y)$$

Since $\gamma_u(t, t\omega) \in \hat{\mathcal{G}}$, it follows:

$$\forall t,\ \gamma_u(t, t\omega) \in \hat{\mathcal{G}} \overset{density}{\Longrightarrow} \forall (t, \theta),\ \gamma_u(t, \theta) \in \hat{\mathcal{G}} \Longrightarrow \forall t,\ \gamma_u(t, \mathbf{0}) \in \hat{\mathcal{G}}$$

so that

$$\bar{\beta} = \left. \frac{d\gamma(t, \mathbf{0})}{dt} \right|_{t=0} \in \hat{\mathfrak{g}}$$

Theorem

1. *The quasi-stroboscopic averaged equation can be rewritten as:*

$$\frac{d}{dt}Y = \sum_{r=1}^{r} \sum_{\mathbf{k}_1,\dots,\mathbf{k}_r \in \mathbb{Z}^d} \frac{\varepsilon^r}{r} \bar{\beta}_{\mathbf{k}_1 \cdots \mathbf{k}_r} [[\dots[f_{\mathbf{k}_1}, f_{\mathbf{k}_2}], f_{\mathbf{k}_3}], \dots], f_{\mathbf{k}_r}](Y)$$

2. *If the original vector field $f(y, t\omega)$ is divergence free, then so is the quasi-stroboscopic averaged vector field.*
3. *If the original vector field $f(y, t\omega)$ is Hamiltonian with $f_{\mathbf{k}} = J^{-1}\nabla H_{\mathbf{k}}$, then so is the quasi-stroboscopic averaged vector field with Hamiltonian*

$$\bar{H} = \sum_{r=1}^{r} \sum_{\mathbf{k}_1,\dots,\mathbf{k}_r \in \mathbb{Z}^d} \frac{\varepsilon^r}{r} \bar{\beta}_{\mathbf{k}_1 \cdots \mathbf{k}_r} \{\{\dots\{H_{\mathbf{k}_1}, H_{\mathbf{k}_2}\}, H_{\mathbf{k}_3}\}, \dots\}, H_{\mathbf{k}_r}\}$$

Outline

Ongoing work:

- Stroboscopic averaging for the nonlinear Schrödinger equation (Castella, C. , Méhats and Murua)
- Stroboscopic symplectic composition methods (C., Murua, Wang)

Perspectives:

- Error estimates
- Quasi-stroboscopic numerical methods
- Extension to the wave equation

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THANK YOU FOR YOUR ATTENTION