Numerical stroboscopic averaging for ODEs and DAEs

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I. HIGHLY OSCILLATORY PROBLEMS

• Consider the oscillatory IVP

$$\frac{dy}{dt} = f(y, \frac{t}{\epsilon}; \epsilon), \quad t_0 \le t \le t_0 + L, \qquad y(t_0) = y_0 \in \mathcal{R}^d,$$

where $f(y, \tau; \epsilon)$ is 2π-periodic in $\tau = t/\epsilon$. (ie f is $2\pi\epsilon$-prdc in t).

• We are interested in the case $\epsilon \ll 1$, $L = \mathcal{O}(1)$ (solution computed over many periods). Direct numerical solution may be very costly.

• In some applications and for the analysis, system may appear in re-scaled format:

$$\frac{dy}{d\tau} = \epsilon f(y, \tau; \epsilon)$$

with integration interval of length L/ϵ.

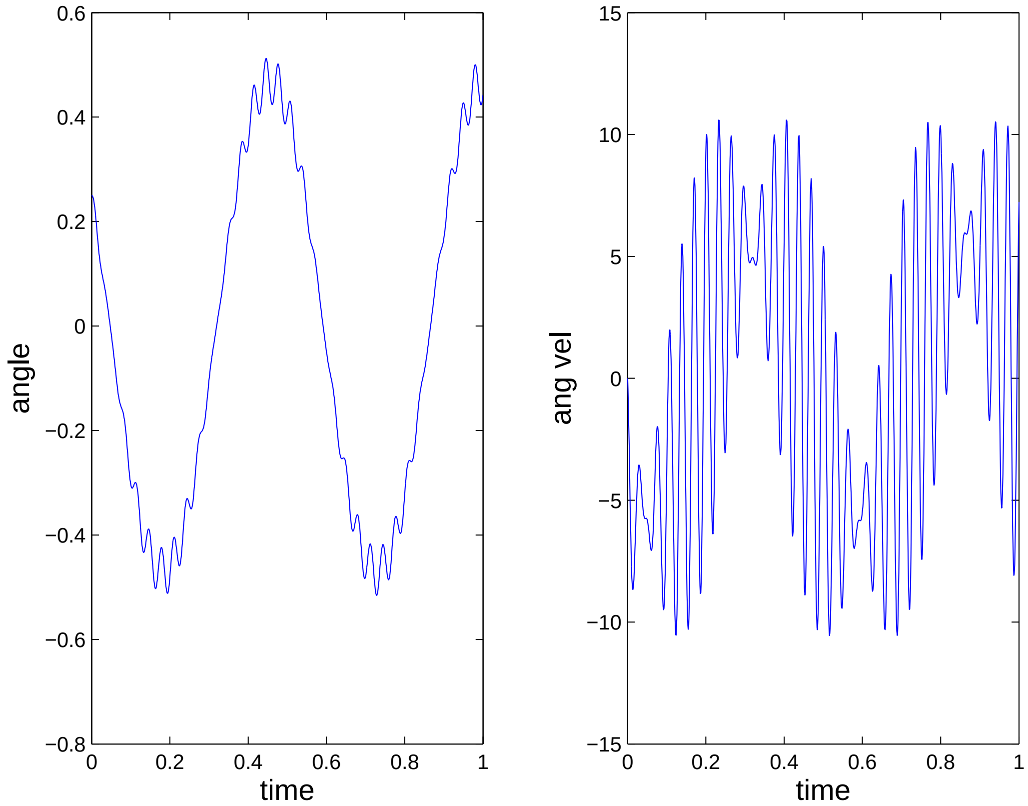
• Denote by $\varphi_{t_0,t}$ the solution operator $y_0 \mapsto y(t)$. Note dependence on t_0 and t (system is not autonomous). It satisfies the property

$$\varphi_{t_1,t_2} \circ \varphi_{t_0,t_1} = \varphi_{t_0,t_2}.$$

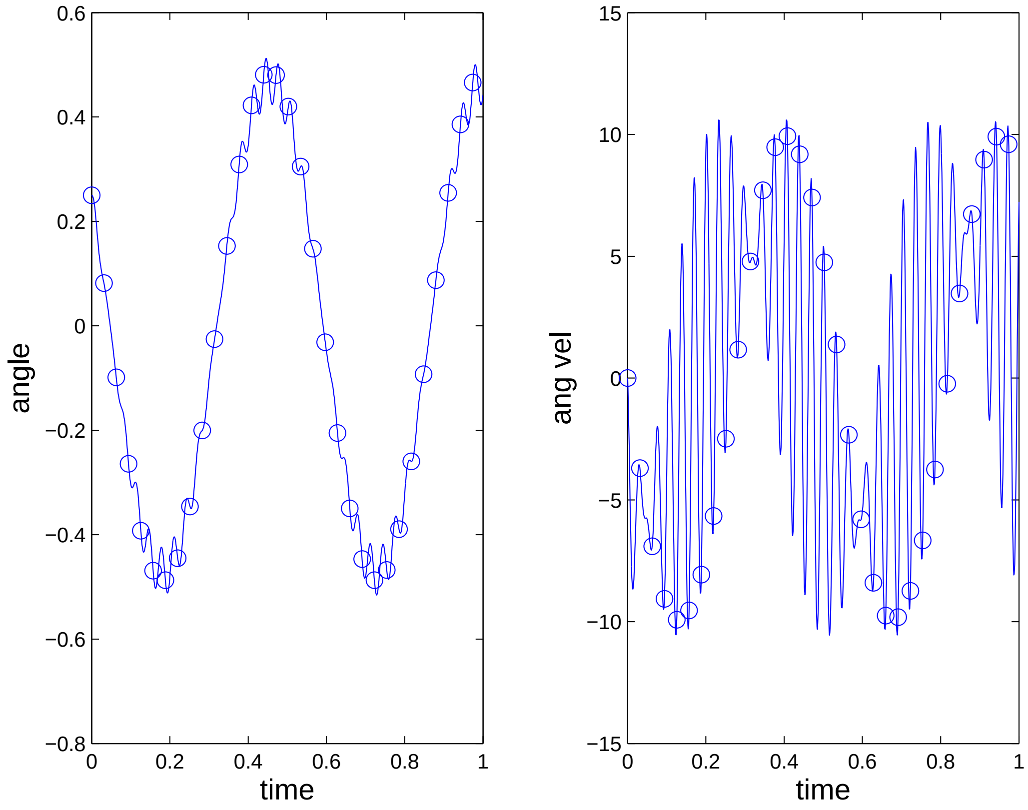
• $\Psi_{t_0} = \varphi_{t_0,t_0+2\pi\epsilon}$ is the one-period or Poincaré map. Its n-th power satisfies $\Psi_{t_0}^n = \varphi_{t_0,t_0+2\pi n\epsilon}$, ie advances the solution over n periods starting from $t = t_0$.

• Attention restricted to cases where $f = \mathcal{O}(1/\epsilon)$ and Ψ_{t_0} is an $\mathcal{O}(\epsilon)$ perturbation of the identity as $\epsilon \downarrow 0$.

• Next slide shows two situations covered by our approach.



• Left: $f = \mathcal{O}(1)$. Solution undergoes $\mathcal{O}(\epsilon)$ changes along one period of length $\mathcal{O}(\epsilon)$. Right: $f = \mathcal{O}(1/\epsilon)$. Solution changes along one period are $\mathcal{O}(1)$ but $\Psi_{t_0} = Id + \mathcal{O}(\epsilon)$



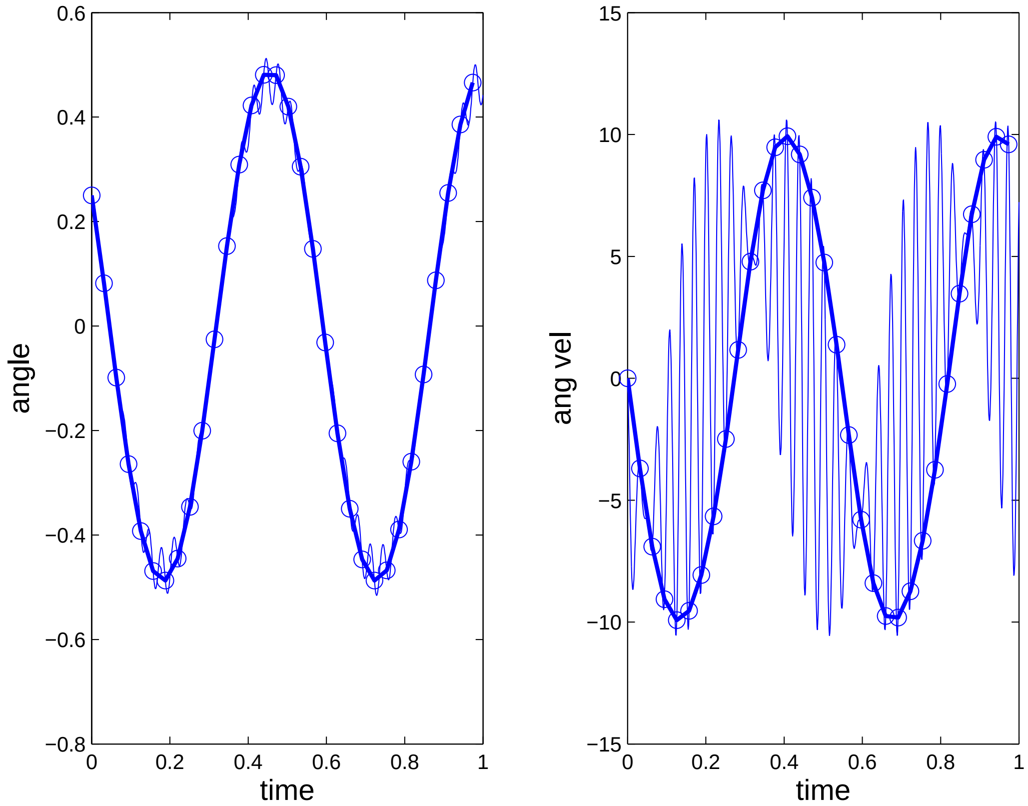
- Changes in solution when t is increased by $2\pi\epsilon$

II. STROBOSCOPIC AVERAGING

• Method of (analytic) averaging. Directly applicable only to situations as in left picture. Try to describe 'smooth' evolution of the system without tracking the fast, period $\mathcal{O}(\epsilon)$, oscillations of true solution $y(t)$.

• $y(t)$ approximated by a 'smooth' $Y(t)$. Usually Y is understood as *average* of y over one period of the fast oscillations.

• Here we look at true solution y with a stroboscopic light that flashes every $2\pi\epsilon$ units of time. Both 'left' and 'right' situations covered:



• Stroboscopic samples $y(t_0)$, $y(t_0 + 2\pi\epsilon)$, $y(t_0 + 4\pi\epsilon)$,... of y (circles) appear to come from 'smooth' function $Y(t)$. Which $Y(t)$?

• Since $\Psi_{t_0} = Id + \mathcal{O}(\epsilon)$, there exist an autonomous modified eqn. $(d/dt)Y = F_\epsilon(Y)$, with t-flow $\Phi_t^{(\epsilon)}$, sch tht $\Psi_{t_0} = \varphi_{t_0,t_0+2\pi\epsilon}$ coincides (formally) with $\Phi_{2\pi\epsilon}^{(\epsilon)}$.

• Hence the n-th power $\Psi_{t_0}^n$ (map that advances y over n periods) coincides with the n-th power of $\Phi_{2\pi\epsilon}^{(\epsilon)}$ ie with $\Phi_{2\pi n\epsilon}^{(\epsilon)}$.

• *Conclusion:* the values

$$y(t_0), \quad y(t_0 + 2\pi\epsilon), \quad \ldots \quad y(t_0 + 2\pi n\epsilon), \quad \ldots$$

of the highly oscillatory solution of $(d/dt)y = f(y, t/\epsilon; \epsilon)$ coincide with the values

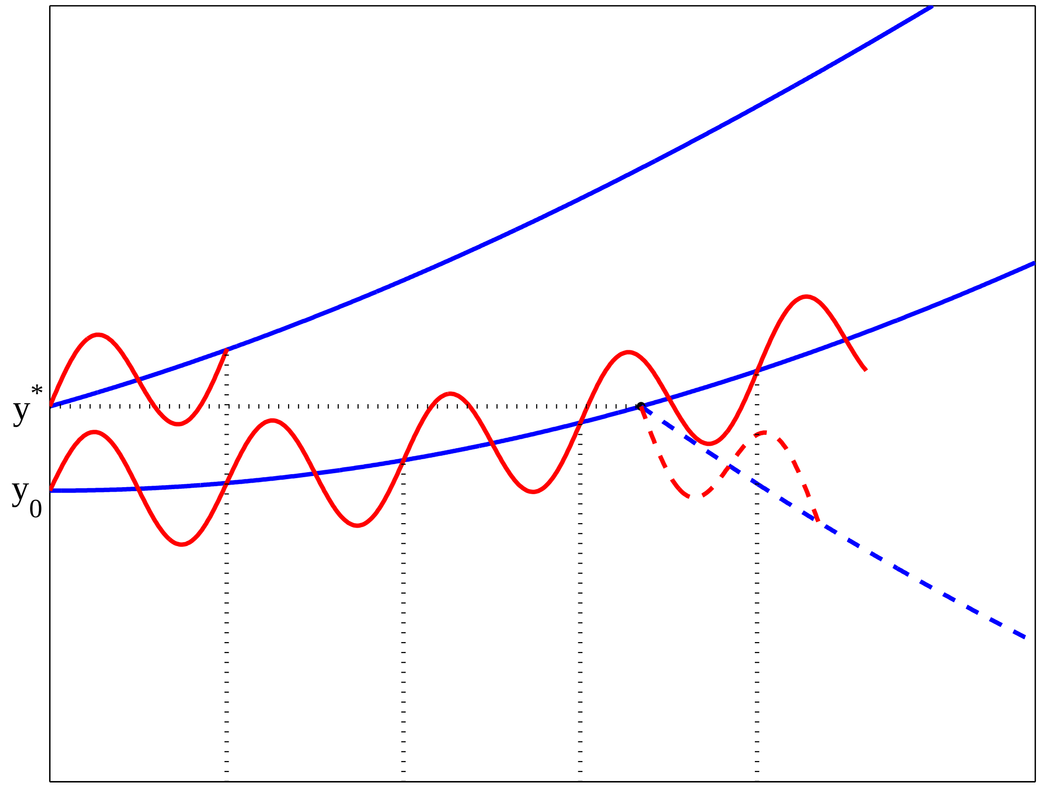
$$Y(t_0), \quad Y(t_0 + 2\pi\epsilon), \quad \ldots \quad Y(t_0 + 2\pi n\epsilon), \quad \ldots$$

of the solution of $(d/dt)Y = F_\epsilon(Y)$ such that $Y(t_0) = y(t_0)$.

Two remarks:

• Coincidence is as formal power series in ϵ. Truncating the formal series of the 'exact' F_ϵ, one obtains averaged systems with $O(\epsilon)$, $O(\epsilon^2)$, ... errors. These issues are ignored in presentation.

• If the initial condition were prescribed at a different value of t_0, then the Poincaré operator $y_0 \mapsto y(t_0 + 2\pi\epsilon)$ changes and one obtains a *different* F_ϵ. (Broken lines in next figure.)



Red wiggly lines: solutions of ivp's corresponding to two initial conditions, y_0 and $y*$ imposed at $t = t_0$. Solid blue lines: solutions of $(d/dt)Y = F_\epsilon(Y)$ with same initial data.

Chartier, Murua, SS, FoCM 2010 show:

• Possible to find systematically the explicit analytic expression for F_ϵ in terms of f by using ideas from the modern analysis of numerical methods —trees, B-series, . . . —.

• Such an explicit expression is useful on its own right to obtain analytically averaged system of high order of accuracy and to systematized the method of averaging.

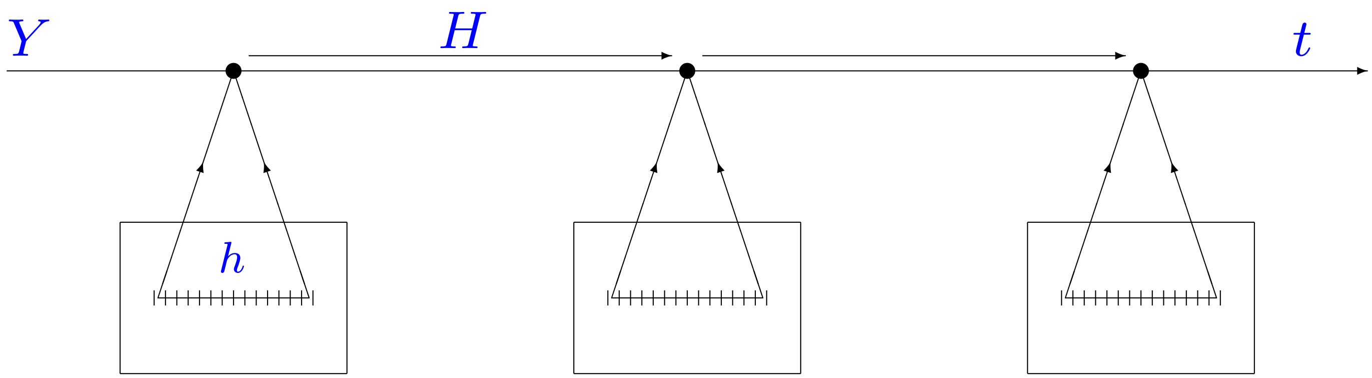
• Furthermore, may be used to analyze numerical methods . . . (idea not pursued here).

III. SAM: A NUMERICAL METHOD BASED ON STROBOSCOPIC AVERAGING

• We shall compute the smooth interpolant $Y(t)$ by integrating the averaged equation $dY/dt = F_\epsilon(Y)$ with a numerical method (macro-solver) with macro-step size H (much) larger than the fast period $2\pi\epsilon$.

• In the spirit of the Heterogeneous Multiscale Methods of E and Engquist, our algorithm does not require the explicit knowledge of the analytic form of F_ϵ. Info. on F_ϵ is gathered on the fly by integrating [with micro-step size h] the original system $dy/dt = f$ in small time-windows of length $\mathcal{O}(\epsilon)$.

• There is much freedom in the choice of the macro-solver and micro-solver, including standard variable-step/order codes.


Y
H
t
h

- How to compute F_ϵ at a given value Y^* of its argument?

- Recall that the t-flow of the vector field F_ϵ is $\Phi_t^{(\epsilon)}$:

$$F_\epsilon(Y^*) = \frac{d}{dt}\Phi_t^{(\epsilon)}(Y^*)\Big|_{t=t_0}.$$

- In algorithm, derivative approximated by differences, such as

$$F_\epsilon(Y^*) = \frac{1}{2\delta}[\Phi_\delta^{(\epsilon)}(Y^*) - \Phi_{-\delta}^{(\epsilon)}(Y^*)] + O(\delta^2).$$

- Choosing $\delta = 2\pi\epsilon$, results in $\Phi_{\pm\delta}^{(\epsilon)} = \varphi_{t_0,t_0\pm\delta}$ (stroboscopic effect) and

$$F_\epsilon(Y^*) \approx (1/(4\pi\epsilon))[\varphi_{t_0,t_0+2\pi\epsilon}(Y^*) - \varphi_{t_0,t_0-2\pi\epsilon}(Y^*)].$$

• $\varphi_{t_0, t_0 \pm 2\pi\epsilon}(Y^*)$ computed by solving the originally given $dy/dt = f(y, t/\epsilon; \epsilon)$, over $t_0 - 2\pi\epsilon \leq t \leq t_0 + 2\pi\epsilon$, with initial condition $y(t_0) = Y^*$.

• Of course, one may use other finite-difference formulae such as the fourth-order based on $t_0 + 2\pi k\epsilon$, $k = 0, \pm 1, \pm 2$.

• Note lack of synchrony between macro and micro integrations. Micro-integration always start from t_0. Starting micro-integratns from current value of t in macro-integration will not do: refer to preceding figure.

• Algorithm presented evolved from our study of Heterogeneous Multiscale Method (E, Engquist, Tsai, Sharp, Ariel, . . .)

• Basic underlying idea has appeared several times in the literature over the last fifty years (in particular, in astronomy and circuit theory): envelope-following methods, multirevolution methods, . . . Taratynova, Mace and Thomas, Graff and Bettis, Gear/Petzold/Gallivan,. . . (outer integrator has to be built on purpose).

• Kirchgraber 1982, 1988 uses high-order RKs. Recovery of macro-field not from numerical differentiation.

• For comparison refer to written version of present talk.

`http://hermite.mac.cie.uva.es/sanzserna`

IIII. ERROR ANALYSIS

- Three sources of errors:

1. Approximate true values of F_ϵ by a finite difference approximation $\widetilde{F}_\epsilon$. Error is $\mathcal{O}(\epsilon^2)$ for 2nd order differencing.

2. Use in difference formula of $\varphi_{t_0,t_0\pm 2\pi\epsilon}(Y^*)$ obtained via micro-integration. Error in $\varphi_{t_0,t_0\pm 2\pi\epsilon}(Y^*)$ is $\mathcal{O}((\Delta\tau)^p) = \mathcal{O}((h/\epsilon)^p)$, where p is the order of the micro-integrator. Errors in F_ϵ are then $\mathcal{O}(\epsilon^{-1}(h/\epsilon)^p)$.

3. Use of macro-integrator to solve averaged equation. Error $\mathcal{O}(H^P)$, where P is the order of the macro-integrator.

- Summing up

$$\mathcal{O}\left(\epsilon^2 + H^P + \frac{1}{\epsilon}\left(\frac{h}{\epsilon}\right)^p\right) = \mathcal{O}\left(\epsilon^2 + H^P + \frac{1}{\epsilon}(\Delta\tau)^p\right),$$

- In some cases, the micro-integration error is $\mathcal{O}(\epsilon^\nu(\Delta\tau)^p)$ with $\nu > 0$ (ie errors vanish if $\epsilon \downarrow 0$ with h fixed). Then we have

$$\mathcal{O}\left(\epsilon^2 + H^P + \epsilon^{\nu-1}\left(\frac{h}{\epsilon}\right)^p\right) = \mathcal{O}\left(\epsilon^2 + H^P + \epsilon^{\nu-1}(\Delta\tau)^p\right).$$

V. NUMERICAL RESULTS

(A) A perturbed Kepler problem in the plane (from Kirchgraber):

$$\frac{d}{ds}x = v, \quad \frac{d}{ds}v = -\frac{1}{r^3}x + \epsilon G(x),$$

where

$$G(x) = -\nabla V(x), \quad V(x) = -\frac{1}{2r^3} + \frac{3x_1^2}{2r^5}, \quad r = \sqrt{x_1^2 + x_2^2}.$$

Use fictitious time $\tau = \lambda(x,v)s$, with $\lambda(x,v) = (-2E(x,v))^{-3/2}$ (E denotes energy), and system becomes

$$\frac{d}{d\tau}x = \lambda(x,v)v, \quad \frac{d}{d\tau}v = \lambda(x,v)\left(-\frac{1}{r^3}x + \epsilon G(x)\right).$$

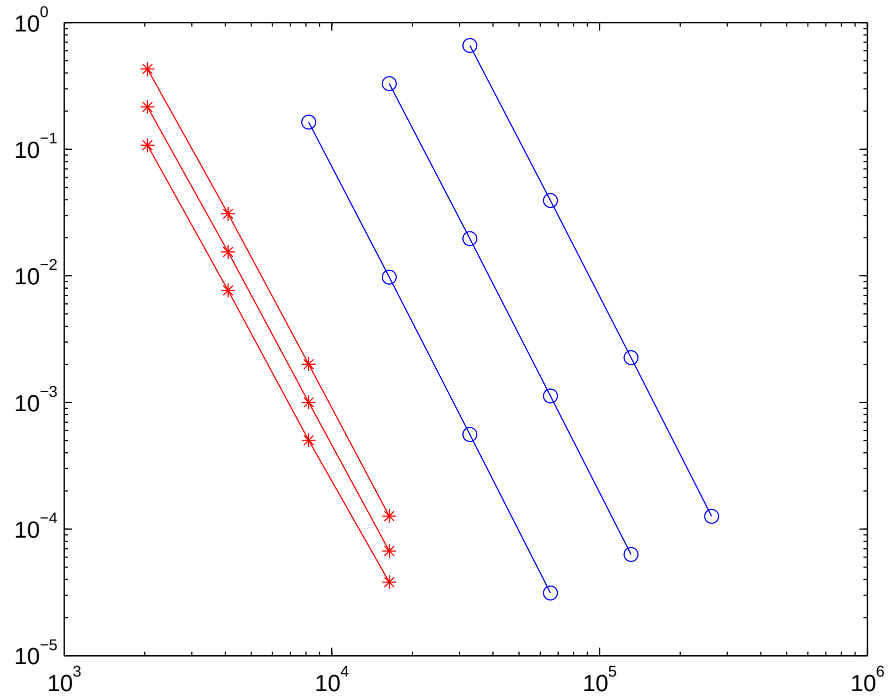
If $\epsilon = 0$ (unperturbed) all solutions are 2π-periodic in τ.

- $x_1(0) = 1$, $x_2(0) = 0$, $v_1(0) = 0$, $v_2(0) = 1$.

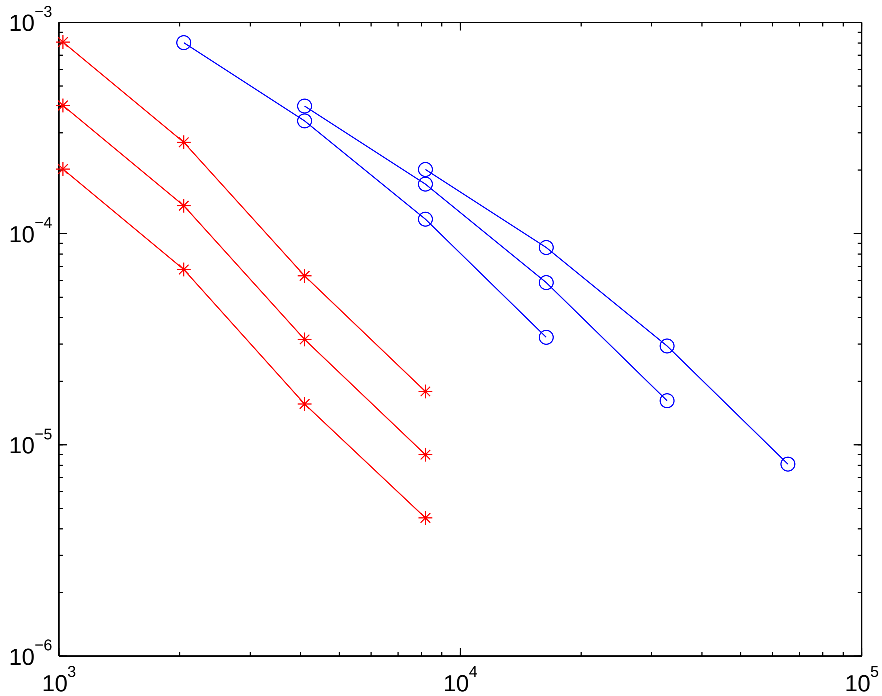
- $\epsilon = 2^{-12}, 2^{-13}, 2^{-14}$ ($2^{-12} \approx 2.4 \times 10^{-4}$).

- Integration interval $0 \leq \tau \leq (\pi/8)\epsilon^{-1}$.

- Constant-step classical RK4 as macro-integrator. Second-order differences.



(A1) Error vs. number of micro-steps, stars: SAM with RK4 micro-integrator 8 macro-steps, circles: standard RK4. Halving ϵ doubles the error



(A2) Error vs. number of micro-steps, stars: SAM with (Strang like) splitting (Kepler+perturbation) micro-integrator 16 macro-steps, circles: standard splitting. Halving ϵ halves the error ($\nu = 2$).

Summary: When $\Delta\tau$ is kept fixed and ϵ is halved:

- The standard RK integrator works twice as much and doubles the error.
- The standard splitting scheme works twice as much and halves the error.
- SAM with RK micro-integrations uses the same work and doubles the error.
- SAM with splitting micro-integration uses the same work and halves the error.

(B) Van der Pol:

$$\frac{d}{d\tau}q = p, \qquad \frac{d}{d\tau}p = -q + \epsilon(1 - q^2)p.$$

Perturbed harmonic oscillator. When the initial condition is away from limit cycle, solution needs $\mathcal{O}(1/\epsilon)$ time-interval to reach the limit-cycle. In transient phase, solution changes by $\mathcal{O}(\epsilon)$ between consecutive stroboscopic times. Near limit cycle by $\mathcal{O}(\epsilon^2)$.

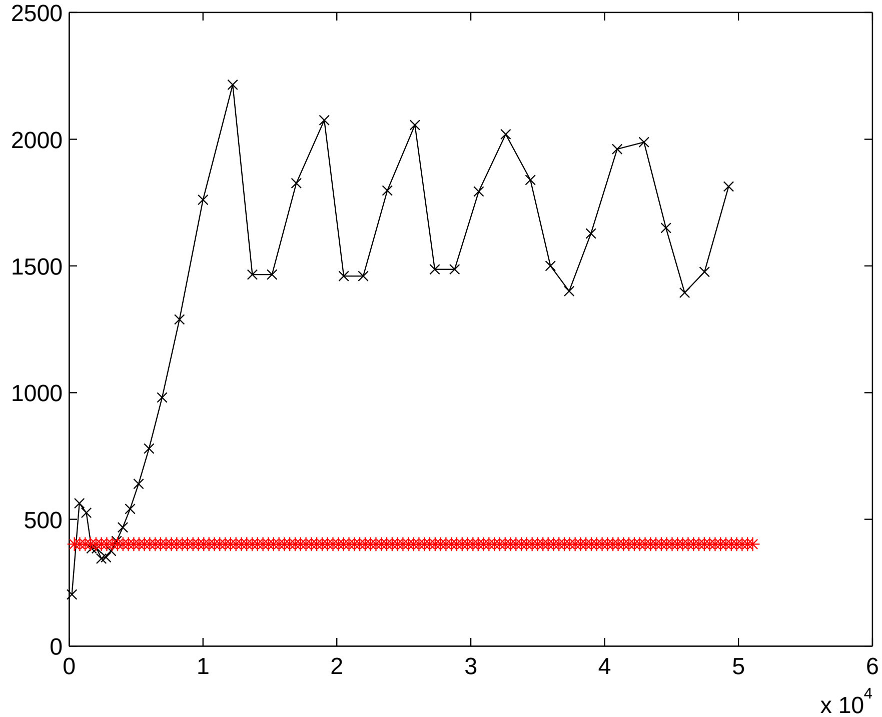
- $q(0) = p(0) = 0.5$, $\epsilon = 2^{-9}$, $0 \le \tau \le \tau_{\text{end}} = 32\pi\epsilon^{-1} \approx 51,000$

The following runs yield roughly the same errors:

- SAM with (variable step-size) ode45 macro-integrator (40 macro-steps); Strang splitting micro-integration $\Delta\tau = \pi/16$

- SAM with the fifth-order formula of ode45, constant step-size (128 macro-steps); Strang splitting micro-integration $\Delta\tau = \pi/16$

- Strang-splitting (260,000 steps), $\Delta\tau = \pi/16$



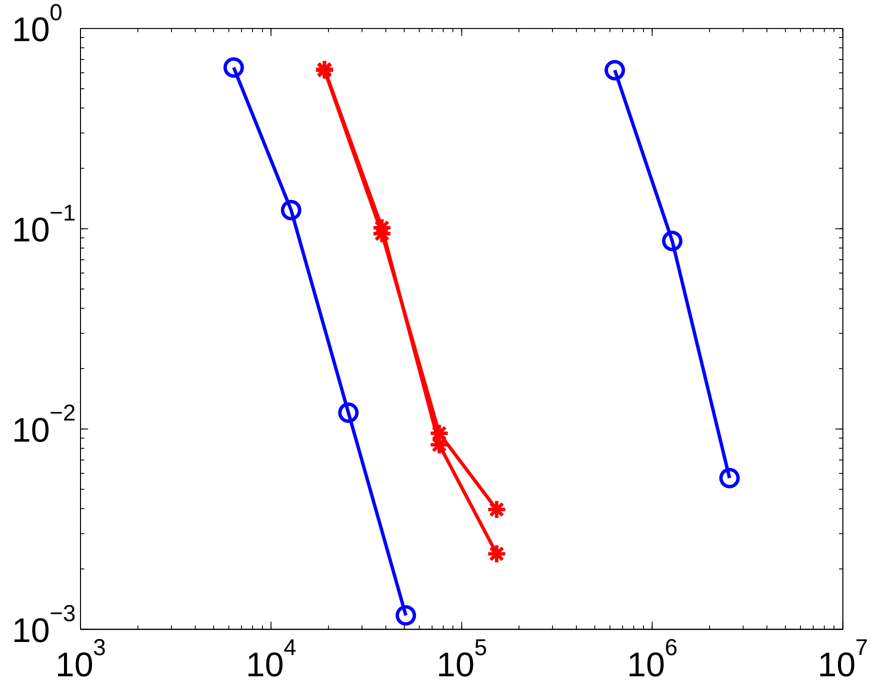
• SAM: macro-step-length in ode45 as a function of τ and macro-step-length in constant step-size implementation. Note H may be $2,000$ or larger!

(C) DAEs:

- Approach easily extended to DAEs.

- Eg: vibrated inverted pendulum and vibrated double inverted pendulum formulated in cartesian coordinates. (Index 2 DAEs, if GGL approach used.)

- Half-explicit RK method of order 3 (Brasey/Hairer (1993)) successfully implemented.



• Error vs. number of micro-steps, $\epsilon = 10^{-4}$, 10^{-6}, stars: SAM with macro-step-size $H = \pi/2500$, circles: standard integration ($h = 2\pi\epsilon/n, n = 2^j, j = 2, 3, \ldots$). Dividing ϵ by 100 does not change the error ($\nu = 1$).