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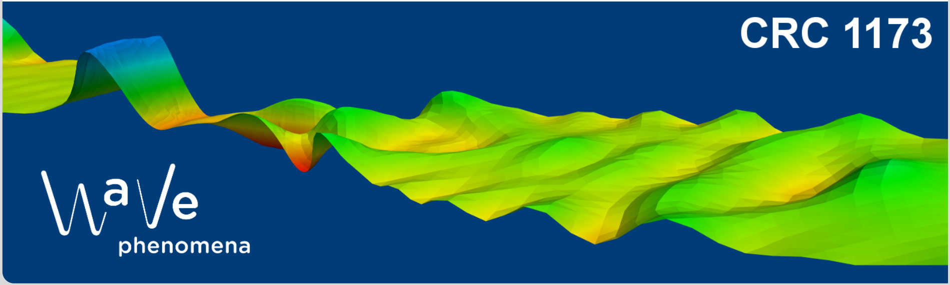
Numerical solution of the wave equation with acoustic boundary conditions

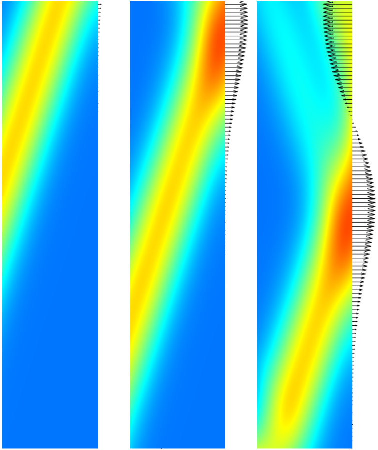
David Hipp

joint work with Marlis Hochbruck

Numerical Analysis Group



Introduction



example

- design of concert hall
- model walls as tiny springs
- springs react to excess pressure

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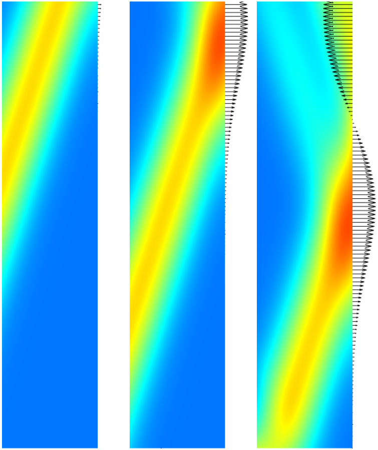

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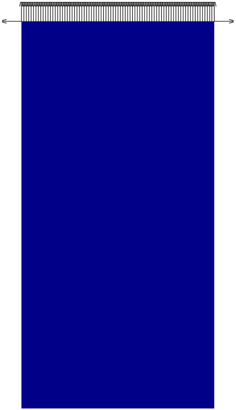
setting

- bounded domain $\Omega \subset \mathbb{R}^2$
- boundary $\Gamma = \partial\Omega$
- u velocity potential
- δ infinitesimal normal displ. of Γ
- Γ not moving!

Acoustic boundary conditions

introduced in [Beale, Rosencrans ’74]

- $u\colon [0,T] \times \Omega \to \mathbb{R}$
- $\delta\colon [0,T] \times \Gamma \to \mathbb{R}$



wave equation with acoustic b.c.

u and δ are governed by:

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\begin{aligned}
\partial_t^2 u &= c^2 \Delta u && \text{in } \Omega \\
m\partial_t^2 \delta + k\delta + d\delta &= -\rho \partial_t u && \text{on } \Gamma \\
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$+$ initial conditions for $u, \partial_t u, \delta, \partial_t \delta$

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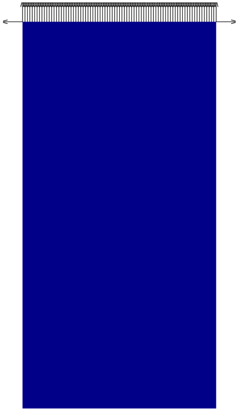
- related to Wentzell boundary condition [Gal, Goldstein, Goldstein '03]



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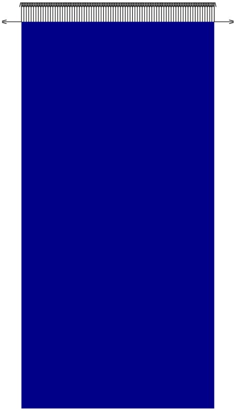

- related to Wentzell boundary condition [Gal, Goldstein, Goldstein '03]
- dynamic boundary condition

Boundary conditions for the wave equation

- energies of u

kinetic $KE(u) = \frac{1}{2} \int_\Omega \frac{1}{c^2} |\partial_t u|^2 \, \mathrm{d}x$

potential $PE(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x$

[G.R. Goldstein '06] for details

Boundary conditions for the wave equation

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- minimizing action functional $S(u) = \int_0^T KE(u) - PE(u)\,\mathrm{d}t$

$$\partial_t^2 u = c^2 \Delta u \qquad \text{in } \Omega$$

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- kinetic boundary conditions
- Robin and Neumann b.c. are contained
- dynamic boundary condition for $\lambda > 0$

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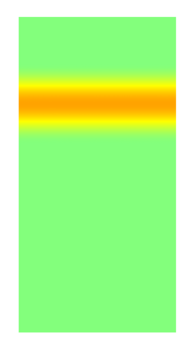
Comparison of different boundary conditions

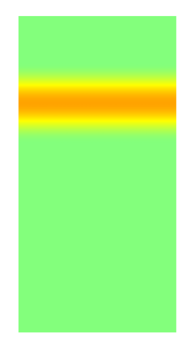
Neumann

$$\partial_\nu u = 0$$



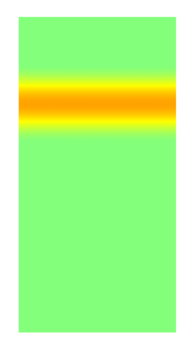
Robin

$$\partial_\nu u + u = 0$$



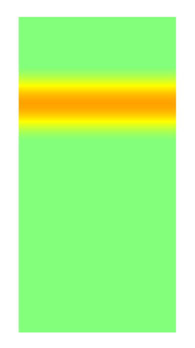
kinetic

$$\partial_t^2 u + ku = -\partial_\nu u$$



acoustic

$$\partial_t^2 \delta + k\delta = -\partial_t u$$



Analysis I

well-posedness of wave equation

$$\partial_t^2 u = \Delta u \quad \text{in } \Omega \qquad + \quad \text{i.c.}$$

with Neumann b.c.

$$\partial_\nu u = 0 \qquad \text{on } \Gamma$$

with acoustic b.c.

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with acoustic b.c.

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as 1st order evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{u}(t) = A\vec{u}(t) \qquad + \quad \text{i.c.}$$

with operator

$$A\vec{u} = \begin{bmatrix} 0 & \mathrm{I} \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \qquad \mathcal{A}\vec{u} = \begin{bmatrix} u_t \\ \Delta u \\ \delta_t \\ -\gamma(u_t) - k\delta \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} u \\ u_t \\ \delta \\ \delta_t \end{bmatrix}$$

Analysis II

- (energy) Hilbert space

$$H = H^1_c(\Omega) \times L^2(\Omega) \times L^2(\Gamma) \times L^2(\Gamma) = \mathcal{H}$$

$$\|\vec{u}\|^2_H = \frac{1}{2}\int_\Omega |\nabla u|^2 + |u_t|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Gamma k\,|\delta|^2 + |\delta_t|^2 \,\mathrm{d}\sigma = \|\vec{u}\|^2_{\mathcal{H}}$$

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- domain of operator

$$\mathcal{D}(A) = \{\vec{u} \in H \mid \Delta u \in L^2(\Omega),\ u_t \in H^1(\Omega),\ \partial_\nu u = 0 \text{ on } \Gamma\}$$

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Analysis II

- (energy) Hilbert space

$$H = H_c^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma) \times L^2(\Gamma) = \mathcal{H}$$

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Theorem ([Beale '76, Thm. 2.1])

- *$\mathcal{A}$ closed, densely defined and skewadjoint in $\mathcal{H}$*
- $$\frac{\mathrm{d}}{\mathrm{d}t}\vec{u}(t) = \mathcal{A}\vec{u}(t), \qquad \vec{u}(0) = \vec{u}_0 \in \mathcal{D}(\mathcal{A})$$

  is well-posed and $\|\vec{u}\|_{\mathcal{H}} = const.$

Numerical discretization

method of lines

1. spatial discretization
2. numerical time integration of stiff ODE

Numerical discretization

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method of lines

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2. numerical time integration of stiff ODE

- **difficulty:** discretization of $H_c^1(\Omega) = H^1(\Omega)/\mathbb{R}$

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- **solution:** choose $\mathcal{H} = H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma) \times L^2(\Gamma)$

$$\|\vec{u}\|^2_{\mathcal{H}} = \int_\Omega |u|^2 \,\mathrm{d}x + \|\vec{u}\|^2_{\mathcal{H}_c}$$

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$$\|\vec{u}\|^2_{\mathcal{H}} = \int_\Omega |u|^2 \,\mathrm{d}x + \|\vec{u}\|^2_{\mathcal{H}_c}$$

- $\mathcal{A}\colon \mathcal{H} \supset \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ not skewadjoint, but

$$\left(\vec{v} \,\middle|\, \left(\mathcal{A} - \tfrac{1}{2}\,\mathrm{I}\right)\vec{v}\right)_{\mathcal{H}} \leq 0, \qquad \vec{v} \in \mathcal{D}(\mathcal{A})$$

Corollary

$\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is the infinitesimal generator of a C_0-semigroup

Variational formulation

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Green's formula: $\Delta u \in L^2(\Omega)$, $v_t \in H^1(\Omega)$, $\partial_\nu u = \delta_t$

$$\left(\vec{v} \,\middle|\, \mathcal{A}\vec{u}\right)_{\mathcal{H}} = (v \,|\, u_t)_{1,\Omega} + (v_t \,|\, \Delta u)_{0,\Omega} + (k\eta \,|\, \delta_t)_{0,\Gamma} - (\eta_t \,|\, \gamma u_t + k\delta)_{0,\Gamma}$$

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variational problem

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Spatial discretization

- $V_h^\Omega \subset H^1(\Omega)$ and $V_h^\Gamma \subset L^2(\Gamma)$ finite dim. subspaces with $\gamma(V_h^\Omega) \subset V_h^\Gamma$

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semidiscrete problem

find $\vec{u}_h \colon [0, T] \to \mathcal{V}_h$ s.t.

$$\Big(\vec{v}_h \,\big|\, \tfrac{\mathrm{d}}{\mathrm{d}t}\vec{u}_h(t)\Big)_{\mathcal{H}} = s\big(\vec{v}_h, \vec{u}_h(t)\big) \qquad \forall \vec{v}_h \in \mathcal{V}_h \qquad + \text{ i.c. in } \mathcal{V}_h$$

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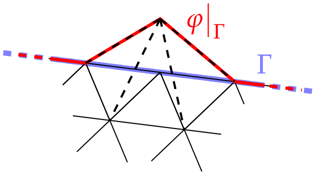
finite element spaces

- $\mathcal{T}_h$ regular triangulation of Ω
- V_h^Ω = pcw. linear FEs over $\mathcal{T}_h$
- $V_h^\Gamma = \gamma\left(V_h^\Omega\right)$



Convergence FE-semidiscretization I

1. split error with orthogonal projection $\mathcal{P}_h \colon \mathcal{H} \to \mathcal{V}_h$

$$\vec{e} = \vec{u}_h - \vec{u} = \left(\vec{u}_h - \mathcal{P}_h\vec{u}\right) + \left(\mathcal{P}_h\vec{u} - \vec{u}\right) = \vec{e}_h + \vec{e}_{\mathcal{P}}$$

$\left(\mathcal{P}_h\vec{u} - \vec{u} \,\middle|\, \vec{v}_h\right)_{\mathcal{H}} = 0$ for all $\vec{v}_h \in \mathcal{V}_h$

Convergence FE-semidiscretization I

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$$\vec{e} = \vec{u}_h - \vec{u} = \big(\vec{u}_h - \mathcal{P}_h\vec{u}\big) + \big(\mathcal{P}_h\vec{u} - \vec{u}\big) = \vec{e}_h + \vec{e}_{\mathcal{P}}$$

2. equation for $\vec{e}_h \in \mathcal{V}_h$

$$\begin{aligned}
\left(\vec{v}_h \,\middle|\, \tfrac{\mathrm{d}}{\mathrm{d}t}\vec{e}_h\right)_{\mathcal{H}} &= \left(\vec{v}_h \,\middle|\, \tfrac{\mathrm{d}}{\mathrm{d}t}\vec{e}_h + \tfrac{\mathrm{d}}{\mathrm{d}t}\vec{e}_{\mathcal{P}}\right)_{\mathcal{H}} \\
&= \left(\vec{v}_h \,\middle|\, \tfrac{\mathrm{d}}{\mathrm{d}t}(\vec{u}_h - \vec{u})\right)_{\mathcal{H}} \\
&= s\big(\vec{v}_h, \vec{u}_h - \vec{u}\big) \\
&= s\big(\vec{v}_h, \vec{e}_h\big) + s\big(\vec{v}_h, \vec{e}_{\mathcal{P}}\big), \qquad \forall \vec{v}_h \in \mathcal{V}_h
\end{aligned}$$

$\big(\mathcal{P}_h\vec{u} - \vec{u} \,\big|\, \vec{v}_h\big)_{\mathcal{H}} = 0$ for all $\vec{v}_h \in \mathcal{V}_h$

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\end{aligned}$$

3. set $\vec{v}_h = \vec{e}_h$

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\vec{e}_h\|_{\mathcal{H}}^2 = s\big(\vec{e}_h, \vec{e}_h\big) + s\big(\vec{e}_h, \vec{e}_{\mathcal{P}}\big) \le C \|\vec{e}_h\|_{\mathcal{H}}^2 + s\big(\vec{e}_h, \vec{e}_{\mathcal{P}}\big)$$

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Convergence FE-semidiscretization II

4. with $\mathcal{P}_h$-properties and standard FE approximation results

$$\left|s(\vec{e}_h, \vec{e}_{\mathcal{P}})\right| \leq C \,\|\vec{e}_h\|_{\mathcal{H}}^2 + C \left(\|P_{0,\Omega} u_t - u_t\|_{1,\Omega}^2 + \|P_{1,\Omega} u - u\|_{0,\Omega}^2 \right)$$
$$\leq C \,\|\vec{e}_h\|_{\mathcal{H}}^2 + \widetilde{C}\big(\,|u_t|_{2,\Omega}, |u|_{2,\Omega} \big) h^2$$

C and $\widetilde{C}$ independent of h

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$$\|\vec{e}_h(t)\|_{\mathcal{H}} \leq Ch$$

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6. apply Gronwall's lemma

$$\|\vec{e}_h(t)\|_{\mathcal{H}} \leq Ch$$

7. consider full error

$$\|\vec{e}(t)\|_{\mathcal{H}} \leq \|\vec{e}_h(t)\|_{\mathcal{H}} + \|\vec{e}_{\mathcal{P}}(t)\|_{\mathcal{H}} \leq Ch$$

Convergence FE-semidiscretization

Theorem (H., Hochbruck 2015)

$$\vec{u} \in L^\infty\Big([0,T]; H^2(\Omega) \times H^1(\Omega) \times H^1(\Gamma) \times H^1(\Gamma)\Big)$$

$$u_t \in L^2\Big([0,T]; H^2(\Omega)\Big)$$

then $$\|\vec{u}_h(t) - \vec{u}(t)\|_{\mathcal{H}} \leq Ch, \qquad t \in [0,T]$$

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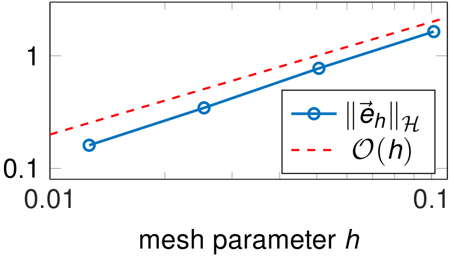

Comment on time integration

method of lines gives stiff problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}_h\mathbf{u}(t) = \mathbf{S}_h\mathbf{u}(t)$$

with exact solution

$$\mathbf{u}(t) = \exp\left(t\,\mathbf{M}_h^{-1}\mathbf{S}_h\right)\mathbf{u}(0), \qquad t \geq 0$$

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   - direct approximation in Krylov subspaces

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1. polynomial approximation
   - explicit time stepping schemes
   - direct approximation in Krylov subspaces
2. rational approximation
   - implicit time stepping schemes
   - direct approximation with rational Krylov methods

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1. polynomial approximation
   - explicit time stepping schemes
   - direct approximation in Krylov subspaces
2. rational approximation
   - implicit time stepping schemes
   - direct approximation with rational Krylov methods
3. combinations of both

Outlook

- evolution equation as boundary conditions, e.g.

$$\begin{aligned} \partial_t^2 u &= c^2 \Delta u && \text{in } \Omega \\ \partial_t^2 \delta + c_\Gamma \Delta_\Gamma \delta &= -\partial_t u && \text{on } \Gamma \\ \partial_t \delta &= \partial_\nu u && \text{on } \Gamma \end{aligned}$$

- non-linear boundary conditions and coupling
- splitting methods for domain and boundary part

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